

THE AFFINE YANGIAN OF \mathfrak{gl}_1 REVISITED

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ABSTRACT. The affine Yangian of \mathfrak{gl}_1 was recently introduced by Maulik-Okounkov in [MO]. In this article, we provide a loop realization of this algebra arising naturally from the geometric actions. The similarity of the representation theories of this algebra and of the quantum toroidal algebra of \mathfrak{gl}_1 is explained by generalizing the results of [GTL] to this setting.

INTRODUCTION

The goal of this note is to clarify the notion of the affine Yangian of \mathfrak{gl}_1 , studied in [MO] from the geometric point of view. We provide an explicit presentation of this algebra and discuss its representation theory. We also explain its relation to the quantum toroidal algebra of \mathfrak{gl}_1 .

This paper is organized in the following way:

- In Section 1, we recall the definition of the algebra $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ and introduce its additive analogue $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$. We also discuss a slightly different presentation of $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$.
- In Section 2, we recall the $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -action on the direct sum of equivariant K-groups of the Hilbert scheme of points on \mathbb{A}^2 following [FT1]. We also formulate a similar result about the $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -action on an analogous sum of equivariant cohomologies.
- In Section 3, we generalize the results of Section 2 to the Gieseker moduli spaces $M(r, n)$.
- In Section 4, we recall some series of $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representations. We introduce their analogues for $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$. We also relate them to the representations from Section 3. We conclude by introducing appropriate categories \mathcal{O} .
- In Section 5, we describe explicitly the limits of both algebras in interest as one of the parameters trivializes, i.e., either $q_3 \rightarrow 1$ or $h_3 \rightarrow 0$.
- In Section 6, we construct a homomorphism $\Upsilon : \check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1) \rightarrow \hat{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$. This is analogous to the relation between a quantum loop algebra $U_q(L\mathfrak{g})$ and a Yangian $Y_{\hbar}(\mathfrak{g})$ of a semisimple Lie algebra \mathfrak{g} , discovered in [GTL].
- In Section 7, we recall the definition of the small shuffle algebra S^m and its commutative subalgebra \mathcal{A}^m , which played a crucial role in [FT1]. We introduce its additive analogue S^a and a corresponding commutative subalgebra \mathcal{A}^a .
- In Section 8, we discuss a “horizontal realization” of $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$. It allows to define a tensor product structure on the whole category \mathcal{O} and also relates \mathcal{A}^m to certain matrix coefficients. We conclude by discussing the properties of natural geometric Whittaker vectors.
- In Appendix, we present main computations.

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1. BASIC DEFINITIONS

In this section we define the algebras $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$.

1.1. The toroidal algebra of \mathfrak{gl}_1 . Let $q_1, q_2, q_3 \in \mathbb{C} \setminus \{1\}$ satisfy $q_1 q_2 q_3 = 1$.

The toroidal algebra of \mathfrak{gl}_1 , denoted $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$, is an associative unital \mathbb{C} -algebra generated by $\{e_i, f_i, \psi_j^\pm, \psi_0^{\pm-1} \mid i \in \mathbb{Z}, j \in \mathbb{Z}_+\}$ ($\mathbb{Z}_+ := \{n \in \mathbb{Z} \mid n \geq 0\}$) with the following defining relations:

$$(T0) \quad \psi_0^\pm \cdot \psi_0^{\pm-1} = \psi_0^{\pm-1} \cdot \psi_0^\pm = 1, \quad [\psi^\pm(z), \psi^\pm(w)] = 0, \quad [\psi^+(z), \psi^-(w)] = 0,$$

$$(T1) \quad e(z)e(w)(z - q_1 w)(z - q_2 w)(z - q_3 w) = -e(w)e(z)(w - q_1 z)(w - q_2 z)(w - q_3 z),$$

$$(T2) \quad f(z)f(w)(w - q_1 z)(w - q_2 z)(w - q_3 z) = -f(w)f(z)(z - q_1 w)(z - q_2 w)(z - q_3 w),$$

$$(T3) \quad [e(z), f(w)] = \frac{\delta(z/w)}{(1 - q_1)(1 - q_2)(1 - q_3)}(\psi^+(w) - \psi^-(z)),$$

$$(T4) \quad \psi^\pm(z)e(w)(z - q_1 w)(z - q_2 w)(z - q_3 w) = -e(w)\psi^\pm(z)(w - q_1 z)(w - q_2 z)(w - q_3 z),$$

$$(T5) \quad \psi^\pm(z)f(w)(w - q_1 z)(w - q_2 z)(w - q_3 z) = -f(w)\psi^\pm(z)(z - q_1 w)(z - q_2 w)(z - q_3 w),$$

$$(T6) \quad \text{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2+1}, e_{i_3-1}]] = 0, \quad \text{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2+1}, f_{i_3-1}]] = 0,$$

where these generating series are defined as follows:

$$e(z) := \sum_{i=-\infty}^{\infty} e_i z^{-i}, \quad f(z) := \sum_{i=-\infty}^{\infty} f_i z^{-i}, \quad \psi^\pm(z) := \sum_{j \geq 0} \psi_j^\pm z^{\mp j}, \quad \delta(z) := \sum_{i=-\infty}^{\infty} z^i.$$

Remark 1.1. (a) The relations (T0)-(T5) should be viewed as collections of termwise relations, which can be recovered by evaluating the coefficients of $z^k w^l$ on both sides of the equalities.

(b) The algebra $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ differs from the Ding-Iohara algebra, considered in [FT1], by an additional relation (T6). However, it is a correct object to consider as will be explained later.

1.2. The elements t_i . We introduce the generators $\{t_i\}$ instead of $\{\psi_j^\pm\}$, similarly to the case of quantum affine algebras. The main advantage is a simplification of (T4)-(T6).

Choose $\{t_j\}_{\pm j > 0} \subset \mathbb{C}[\psi_0^{\pm-1}, \psi_0^\pm, \psi_1^\pm, \dots]$ as the elements satisfying the following identities:

$$\psi^\pm(z) = \psi_0^\pm \cdot \exp \left(\mp \sum_{\pm m > 0} \frac{\beta_m}{m} t_m z^{-m} \right),$$

where $\beta_m := (1 - q_1^m)(1 - q_2^m)(1 - q_3^m)$. We assume $\beta_m \neq 0$, i.e., q_1, q_2, q_3 are not roots of 1.

This choice of t_i is motivated by the following two results.

Proposition 1.1. *The relations (T4, T5) are equivalent to $[\psi_0^\pm, e_j] = 0 = [\psi_0^\pm, f_j]$ together with*

$$(T4t) \quad [t_i, e_j] = e_{i+j} \text{ for } i \neq 0, j \in \mathbb{Z}.$$

$$(T5t) \quad [t_i, f_j] = -f_{i+j} \text{ for } i \neq 0, j \in \mathbb{Z}.$$

The proof of this proposition follows formally from the identity:

$$\ln \left(\frac{(z - q_1^{-1}w)(z - q_2^{-1}w)(z - q_3^{-1}w)}{(z - q_1 w)(z - q_2 w)(z - q_3 w)} \right) = \sum_{m > 0} -\frac{\beta_m}{m} \cdot \frac{w^m}{z^m}.$$

Proposition 1.2. *If relations (T4t, T5t) hold, then (T6) is equivalent to its particular case*

$$(T6t) \quad [e_0, [e_1, e_{-1}]] = 0, \quad [f_0, [f_1, f_{-1}]] = 0.$$

The algebra $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ also satisfies a natural triangular decomposition, see Appendix A.

1.3. The affine Yangian of \mathfrak{gl}_1 . Let $h_1, h_2, h_3 \in \mathbb{C}$ satisfy $h_1 + h_2 + h_3 = 0$.

The affine Yangian of \mathfrak{gl}_1 , denoted $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$, is an associative unital \mathbb{C} -algebra generated by $\{e_j, f_j, \psi_j | j \in \mathbb{Z}_+\}$ with the following defining relations (indices $i, j \in \mathbb{Z}_+$):

- (Y0) $[\psi_i, \psi_j] = 0$,
- (Y1) $([e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}]) + \sigma_2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = \sigma_3\{e_i, e_j\}$,
- (Y2) $([f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}]) + \sigma_2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = -\sigma_3\{f_i, f_j\}$,
- (Y3) $[e_i, f_j] = \psi_{i+j}$,
- (Y4) $([\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}]) + \sigma_2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) = \sigma_3\{\psi_i, e_j\}$,
- (Y4') $[\psi_0, e_j] = 0, [\psi_1, e_j] = 0, [\psi_2, e_j] = 2e_j$,
- (Y5) $([\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}]) + \sigma_2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) = -\sigma_3\{\psi_i, f_j\}$,
- (Y5') $[\psi_0, f_j] = 0, [\psi_1, f_j] = 0, [\psi_2, f_j] = -2f_j$,
- (Y6) For $i_1, i_2, i_3 \in \mathbb{Z}_+$: $\text{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0, \text{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0$,

where $\sigma_1 := h_1 + h_2 + h_3 = 0$, $\sigma_2 := h_1h_2 + h_1h_3 + h_2h_3$, $\sigma_3 := h_1h_2h_3$, $\{a, b\} := ab + ba$.

1.4. Generating series for $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$. Let us introduce the generating series:

$$e(z) := \sum_{j \geq 0} e_j z^{-j-1}, \quad f(z) := \sum_{j \geq 0} f_j z^{-j-1}, \quad \psi(z) := 1 + \sigma_3 \sum_{j \geq 0} \psi_j z^{-j-1}.$$

Define $\check{Y}_{h_1, h_2, h_3}^{\geq 0}(\mathfrak{gl}_1)$ and $\check{Y}_{h_1, h_2, h_3}^{\leq 0}(\mathfrak{gl}_1)$ as the subalgebras of $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ generated by e_j, ψ_j and f_j, ψ_j , respectively. Let us consider the homomorphisms

$$\sigma^+ : \check{Y}_{h_1, h_2, h_3}^{\geq 0}(\mathfrak{gl}_1) \rightarrow \check{Y}_{h_1, h_2, h_3}^{\geq 0}(\mathfrak{gl}_1), \quad \sigma^- : \check{Y}_{h_1, h_2, h_3}^{\leq 0}(\mathfrak{gl}_1) \rightarrow \check{Y}_{h_1, h_2, h_3}^{\leq 0}(\mathfrak{gl}_1)$$

defined on the generators by $\psi_j \mapsto \psi_j$, $e_j \mapsto e_{j+1}$ and $\psi_j \mapsto \psi_j$, $f_j \mapsto f_{j+1}$, respectively. These are well-defined due to the Yangian analogue of Proposition A.1. Let

$$\mu : \check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)^{\otimes 2} \rightarrow \check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$$

be the multiplication. The following result is straightforward:

Proposition 1.3. *Let us introduce $P^a(z, w) := (z - w - h_1)(z - w - h_2)(z - w - h_3)$. Then:*

(a) *The relation (Y0) is equivalent to*

$$[\psi(z), \psi(w)] = 0.$$

(b) *The relation (Y1) is equivalent to*

$$\partial_z^3 \mu(P^a(z, \sigma_{(2)}^+) e(z) \otimes e_j + P^a(\sigma_{(1)}^+, z) e_j \otimes e(z)) = 0 \quad \forall j \in \mathbb{Z}_+.$$

(c) *The relation (Y2) is equivalent to*

$$\partial_z^3 \mu(P^a(\sigma_{(2)}^-, z) f(z) \otimes f_j + P^a(z, \sigma_{(1)}^-) f_j \otimes f(z)) = 0 \quad \forall j \in \mathbb{Z}_+.$$

(d) *The relation (Y3) is equivalent to*

$$\sigma_3 \cdot (w - z)[e(z), f(w)] = \psi(z) - \psi(w).$$

(e) *The relations (Y4)+(Y4') are equivalent to*

$$P^a(z, \sigma^+) \psi(z) e_j + P^a(\sigma^+, z) e_j \psi(z) = 0 \quad \forall j \in \mathbb{Z}_+.$$

(f) *The relations (Y5)+(Y5') are equivalent to*

$$P^a(\sigma^-, z) \psi(z) f_j + P^a(z, \sigma^-) f_j \psi(z) = 0 \quad \forall j \in \mathbb{Z}_+.$$

2. REPRESENTATION THEORY VIA THE HILBERT SCHEME

2.1. Correspondences and fixed points for $(\mathbb{A}^2)^{[n]}$. We set $X = \mathbb{A}^2$ in this section.

Let $X^{[n]}$ be the Hilbert scheme of n points in X . Its \mathbb{C} -points are the codimension n ideals $J \subset \mathbb{C}[x, y]$. Let $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$ be the Nakajima-Grojnowski correspondence. For $i > 0$, the correspondence $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$ consists of all pairs of ideals (J_1, J_2) of $\mathbb{C}[x, y]$ of codimension n , $n+i$ respectively, such that $J_2 \subset J_1$ and the factor J_1/J_2 is supported at a single point. It is known that $P[1]$ is a smooth variety. Let L be the tautological line bundle on $P[1]$ whose fiber at a point $(J_1, J_2) \in P[1]$ equals J_1/J_2 . There are natural projections \mathbf{p}, \mathbf{q} from $P[1]$ to $X^{[n]}$ and $X^{[n+1]}$, correspondingly.

Consider a natural action of $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ on each $X^{[n]}$ induced from the one on X given by the formula $(t_1, t_2)(x, y) = (t_1 \cdot x, t_2 \cdot y)$. The set $(X^{[n]})^{\mathbb{T}}$ of \mathbb{T} -fixed points in $X^{[n]}$ is finite and is in bijection with size n Young diagrams. For a size n Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$, the corresponding ideal $J_\lambda \in (X^{[n]})^{\mathbb{T}}$ is given by $J_\lambda = \mathbb{C}[x, y] \cdot (\mathbb{C}x^{\lambda_1}y^0 \oplus \dots \oplus \mathbb{C}x^{\lambda_k}y^{k-1} \oplus \mathbb{C}y^k)$.

Notation: For a Young diagram λ , let λ^* be the conjugate diagram and define $|\lambda| := \sum \lambda_i$. For a box \square with the coordinates (i, j) , we define $a_\lambda(\square) := \lambda_j - i$, $l_\lambda(\square) := \lambda_i^* - j$. We denote the diagram obtained from λ by adding a box to its j -th row by $\lambda + \square_j$ or simply by $\lambda + j$.

2.2. Geometric $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -action I. We recall the key theorem from [FT1] (see also [SV]).

Let $'M$ be the direct sum of equivariant (complexified) K -groups: $'M = \bigoplus_n K^{\mathbb{T}}(X^{[n]})$. It is a module over $K^{\mathbb{T}}(\text{pt}) = \mathbb{C}[\mathbb{T}] = \mathbb{C}[t_1, t_2]$. We define

$$M := 'M \otimes_{K^{\mathbb{T}}(\text{pt})} \text{Frac}(K^{\mathbb{T}}(\text{pt})) = 'M \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2).$$

It has a natural grading: $M = \bigoplus_n M_n$, $M_n = K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\text{pt})} \text{Frac}(K^{\mathbb{T}}(\text{pt}))$. According to the localization theorem, restriction to the \mathbb{T} -fixed point set induces an isomorphism

$$K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\text{pt})} \text{Frac}(K^{\mathbb{T}}(\text{pt})) \xrightarrow{\sim} K^{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{K^{\mathbb{T}}(\text{pt})} \text{Frac}(K^{\mathbb{T}}(\text{pt})).$$

The structure sheaves $\{\lambda\}$ of the \mathbb{T} -fixed points J_λ (defined in Section 2.1) form a basis in $\bigoplus_n K^{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{K^{\mathbb{T}}(\text{pt})} \text{Frac}(K^{\mathbb{T}}(\text{pt}))$. Since embedding of a point J_λ into $X^{[|\lambda|]}$ is a proper morphism, the direct image in the equivariant K -theory is well defined, and we denote by $[\lambda] \in M$ the direct image of the structure sheaf $\{\lambda\}$. The set $\{[\lambda]\}$ forms a basis of M .

Let \mathfrak{F} be the *tautological vector bundle* on $X^{[n]}$, whose fiber $\mathfrak{F}|_J$ is naturally identified with the quotient $\mathbb{C}[x, y]/J$. Consider the generating series $\mathbf{a}(z)$, $\mathbf{c}(z) \in M(z)$ defined as follows:

$$\mathbf{a}(z) := \Lambda_{-1/z}^\bullet(\mathfrak{F}) = \sum_{i \geq 0} [\Lambda^i(\mathfrak{F})](-1/z)^i,$$

$$\mathbf{c}(z) := \mathbf{a}(zt_1)\mathbf{a}(zt_2)\mathbf{a}(zt_3)\mathbf{a}(zt_1^{-1})^{-1}\mathbf{a}(zt_2^{-1})^{-1}\mathbf{a}(zt_3^{-1})^{-1}, \text{ where } t_3 := t_1^{-1}t_2^{-1}.$$

Finally, we define the linear operators e_i, f_i, ψ_j^\pm ($i \in \mathbb{Z}, j \in \mathbb{Z}_+$) on M :

$$(1) \quad e_i = \mathbf{q}_*(L^{\otimes i} \otimes \mathbf{p}^*) : M_n \rightarrow M_{n+1},$$

$$(2) \quad f_i = \mathbf{p}_*(L^{\otimes(i-1)} \otimes \mathbf{q}^*) : M_n \rightarrow M_{n-1},$$

$$(3) \quad \psi^\pm(z)|_{M_n} = \sum_{r=0}^{\infty} \psi_r^\pm z^{\mp r} := \left(-\frac{1-t_3 z^{-1}}{1-z^{-1}} \mathbf{c}(z) \right)^\pm \in M_n[[z^{\mp 1}]],$$

where $\gamma(z)^\pm$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

Theorem 2.1. *The operators e_i, f_i, ψ_j^\pm , defined in (1)-(3), satisfy the relations (T0)-(T6) with the parameters $q_i = t_i$. This endows M with the structure of $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representation.*

This theorem is proved in [FT1] modulo a straightforward verification of (T6) (see [FFJMM1]).

2.3. Geometric $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -action I. We provide a cohomological analogue of Theorem 2.1.

Let $'V$ be the direct sum of equivariant (complexified) cohomology: $'V = \bigoplus_n H_{\mathbb{T}}^{\bullet}(X^{[n]})$. It is a module over $H_{\mathbb{T}}^{\bullet}(\text{pt}) = \mathbb{C}[\mathfrak{t}] = \mathbb{C}[s_1, s_2]$, where \mathfrak{t} is the Lie algebra of \mathbb{T} . We define

$$V := 'V \otimes_{H_{\mathbb{T}}^{\bullet}(\text{pt})} \text{Frac}(H_{\mathbb{T}}^{\bullet}(\text{pt})) = 'V \otimes_{\mathbb{C}[s_1, s_2]} \mathbb{C}(s_1, s_2).$$

It has a natural grading: $V = \bigoplus_n V_n$, $V_n = H_{\mathbb{T}}^{\bullet}(X^{[n]}) \otimes_{H_{\mathbb{T}}^{\bullet}(\text{pt})} \text{Frac}(H_{\mathbb{T}}^{\bullet}(\text{pt}))$. According to the localization theorem, restriction to the \mathbb{T} -fixed point set induces an isomorphism

$$H_{\mathbb{T}}^{\bullet}(X^{[n]}) \otimes_{H_{\mathbb{T}}^{\bullet}(\text{pt})} \text{Frac}(H_{\mathbb{T}}^{\bullet}(\text{pt})) \xrightarrow{\sim} H_{\mathbb{T}}^{\bullet}((X^{[n]})^{\mathbb{T}}) \otimes_{H_{\mathbb{T}}^{\bullet}(\text{pt})} \text{Frac}(H_{\mathbb{T}}^{\bullet}(\text{pt})).$$

The fundamental cycles $[\lambda]$ of the \mathbb{T} -fixed points J_{λ} form a basis in $\bigoplus_n H_{\mathbb{T}}^{\bullet}((X^{[n]})^{\mathbb{T}}) \otimes_{H_{\mathbb{T}}^{\bullet}(\text{pt})} \text{Frac}(H_{\mathbb{T}}^{\bullet}(\text{pt}))$. Since embedding of a point J_{λ} into $X^{[|\lambda|]}$ is a proper morphism, the direct image in the equivariant cohomology is well defined, and we will denote by $[\lambda] \in V_{|\lambda|}$ the direct image of the fundamental cycle of the point J_{λ} . The set $\{[\lambda]\}$ forms a basis of V .

We introduce the generating series $\mathbf{C}(z) \in V[[z^{-1}]]$ as follows:

$$\mathbf{C}(z) := \left(\frac{\text{ch}(\mathfrak{F}t_1^{-1}, -z^{-1})\text{ch}(\mathfrak{F}t_2^{-1}, -z^{-1})\text{ch}(\mathfrak{F}t_3^{-1}, -z^{-1})}{\text{ch}(\mathfrak{F}t_1, -z^{-1})\text{ch}(\mathfrak{F}t_2, -z^{-1})\text{ch}(\mathfrak{F}t_3, -z^{-1})} \right)^+,$$

where $\text{ch}(F, \bullet)$ denotes the Chern polynomial of F . We also set $s_3 := -s_1 - s_2$.

Finally, we define the linear operators e_j, f_j, ψ_j ($j \in \mathbb{Z}_+$) on V :

$$(1') \quad e_j = \mathbf{q}_*(c_1(L)^j \cdot \mathbf{p}^*) : V_n \rightarrow V_{n+1},$$

$$(2') \quad f_j = \mathbf{p}_*(c_1(L)^j \cdot \mathbf{q}^*) : V_n \rightarrow V_{n-1},$$

$$(3') \quad \psi(z)|_{V_n} = 1 + s_1 s_2 s_3 \sum_{r=0}^{\infty} \psi_r z^{-r-1} := ((1 - s_3/z)\mathbf{C}(z))^+ \in V_n[[z^{-1}]].$$

Theorem 2.2. *The operators e_j, f_j, ψ_j , defined in (1')-(3'), satisfy the relations (Y0)-(Y6) with the parameters $h_i = s_i$. This endows V with the structure of $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representation.*

Let us compute the matrix coefficients of e_j, f_j, ψ_j in the fixed point basis.

Lemma 2.3. *Consider the fixed point basis $\{[\lambda]\}$ of V .*

(a) *The only nonzero matrix coefficients of the operators e_k, f_k are as follows:*

$$e_{k[\lambda-i, \lambda]} = \frac{((\lambda_i - 1)s_1 + (i - 1)s_2)^k}{(s_1 + s_2)((\lambda_1 - \lambda_i + 1)s_1 + (1 - i)s_2)} \cdot \prod_{j \geq 1} \frac{(\lambda_j - \lambda_i + 1)s_1 + (j - i + 1)s_2}{(\lambda_{j+1} - \lambda_i + 1)s_1 + (j - i + 1)s_2},$$

$$f_{k[\lambda+i, \lambda]} = \frac{(\lambda_i s_1 + (i - 1)s_2)^k ((\lambda_i - \lambda_1 + 1)s_1 + i s_2)}{s_1 + s_2} \cdot \prod_{j \geq 1} \frac{(\lambda_i - \lambda_{j+1} + 1)s_1 + (i - j)s_2}{(\lambda_i - \lambda_j + 1)s_1 + (i - j)s_2}.$$

(b) *The eigenvalue of $\psi(z)$ on $[\lambda]$ equals*

$$\left(\left(1 - \frac{s_3}{z} \right) \prod_{\square \in \lambda} \frac{(1 - \frac{\chi(\square) - s_1}{z})(1 - \frac{\chi(\square) - s_2}{z})(1 - \frac{\chi(\square) - s_3}{z})}{(1 - \frac{\chi(\square) + s_1}{z})(1 - \frac{\chi(\square) + s_2}{z})(1 - \frac{\chi(\square) + s_3}{z})} \right)^+,$$

where $\chi(\square_{i,j}) = (i - 1)s_1 + (j - 1)s_2$ for a box $\square_{i,j}$ staying in the j -th row and i -th column.

This lemma is a *cohomological analogue* of [FT1, Lemma 3.1, Proposition 3.1]. Using this result, proof of Theorem 2.2 reduces to a routine verification of the relations (Y0)-(Y6) in the fixed point basis. The only non-trivial relation is actually (Y3). A similar issue in the K -theory case was resolved by [FT1, Lemma 4.1]. We conclude this section by proving an analogous result.

Lemma 2.4. *Let us consider the linear operator $\phi_{i,j} := [e_i, f_j]$ acting on V .*

(a) *The operator $\phi_{i,j}$ is diagonalizable in the fixed point basis $\{[\lambda]\}$ of V .*

(b) *For any Young diagram λ , we have $\phi_{i,j}([\lambda]) = \gamma_{i+j|\lambda} \cdot [\lambda]$, where*

$$\begin{aligned} (\sharp) \quad \gamma_{m|\lambda} &= s_1^{-2} \sum_{i=1}^k y_i^m \prod_{1 \leq j \leq k-1}^{j \neq i} \frac{(y_i - y_j + s_2)(y_j - y_i + s_1 + s_2)}{(y_i - y_j)(y_j - y_i + s_1)} \cdot \frac{y_i + s_1 + (2-k)s_2}{-y_i + (k-1)s_2} \\ &\quad - s_1^{-2} \sum_{i=1}^k (y_i + s_1)^m \prod_{1 \leq j \leq k-1}^{j \neq i} \frac{(y_j - y_i + s_2)(y_i - y_j + s_1 + s_2)}{(y_j - y_i)(y_i - y_j + s_1)} \cdot \frac{y_i + 2s_1 + (2-k)s_2}{-y_i - s_1 + (k-1)s_2}. \end{aligned}$$

Here $y_i := (\lambda_i - 1)s_1 + (i-1)s_2$ and k is a positive integer such that $\lambda_{k-1} = 0$.

(c) *For any Young diagram λ , we have:*

$$\gamma_{0|\lambda} = -1/s_1 s_2, \quad \gamma_{1|\lambda} = 0, \quad \gamma_{2|\lambda} = 2|\lambda|.$$

Proof.

Parts (a) and (b) follow from Lemma 2.3(a) by straightforward calculations.

Let us now prove (c). First we observe that for $m \geq 0$, the expression for $\gamma_{m|\lambda}$ in (\sharp) is a rational function with simple poles at $y_i = y_j$, $y_i = y_j + s_1$, $y_i = (k-1)s_2$, $y_i = -s_1 + (k-1)s_2$. But an easy counting of residues shows that there are actually no poles and the resulting expression is an element of $\mathbb{C}(s_1, s_2)[y_1, y_2, \dots]$. Let us now consider each of the cases $m = 0, 1, 2$.

◦ *Case 1: $m = 0$.*

Since $\gamma_{0|\lambda}$ is a polynomial in y_i of degree ≤ 0 , it should be just an element of $\mathbb{C}(s_1, s_2)$ independent of λ . Evaluating at the empty diagram, we find $\gamma_{0|\lambda} = \gamma_{0|\emptyset} = -1/s_1 s_2$.

◦ *Case 2: $m = 1$.*

First note that $\gamma_{1|\lambda}$ is a polynomial in y_i of degree ≤ 1 . Further for any i_0 the limit of the expression (\sharp) for $r = 1$ as $y_{i_0} \rightarrow \infty$ while y_j are fixed for all $j \neq i_0$, is finite. Thus $\gamma_{1|\lambda}$ is actually a polynomial of degree 0, that is, an element of $\mathbb{C}(s_1, s_2)$ independent of λ . Evaluating at the empty diagram, we find $\gamma_{1|\lambda} = \gamma_{1|\emptyset} = 0$.

◦ *Case 3: $m = 2$.*

Recall that $\gamma_{2|\lambda}$ is a polynomial in y_i of degree ≤ 2 . However, arguments similar to those used in the previous case show that it is a degree ≤ 1 polynomial in y_i over $\mathbb{C}(s_1, s_2)$. Let us compute the principal linear part of this polynomial.

The coefficient of y_{i_0} equals the limit $\lim_{\xi \rightarrow \infty} \frac{1}{\xi} \gamma_{2|\lambda}$ as y_j is fixed for $j \neq i_0$ and $y_{i_0} = \xi \rightarrow \infty$. Formula (\sharp) implies that this limit is equal to $\frac{2}{s_1}$. Therefore, there exists a λ -independent $F(s_1, s_2) \in \mathbb{C}(s_1, s_2)$ such that $\gamma_{2|\lambda} = \frac{2}{s_1}(\tilde{y}_1 + \tilde{y}_2 + \dots) + F(s_1, s_2) = 2|\lambda| + F(s_1, s_2)$, where $\tilde{y}_i = y_i - ((i-1)s_2 - s_1)^1$. Evaluating at the empty Young diagram, we find $F(s_1, s_2) = 0$. The equality $\gamma_{2|\lambda} = 2|\lambda|$ follows. \square

Arguments similar to those from [FT1] prove $\gamma_{m|\lambda} = \psi_{m|\lambda}$ (see also Appendix B).

Remark 2.1. Comparing $(3')$ with Lemma 2.3(b), we find the next ψ -coefficient:

$$\psi_{3|\lambda} = 6 \sum_{\square \in \lambda} \chi(\square) + 2(s_1 + s_2)|\lambda|.$$

In particular, $\frac{1}{6}(\psi_3 + s_3\psi_2)$ corresponds to the cup product with $c_1(\mathfrak{F})$. This operator was first studied by M. Lehn. It is also related to the Laplace-Beltrami operator (see [Na2, Section 4]).

¹ Note that for any Young diagram λ , the sequence $\{\tilde{y}_i\}$ stabilizes to 0 as $i \rightarrow \infty$, unlike $\{y_i\}$.

3. REPRESENTATION THEORY VIA THE GIESEKER SPACE

The Hilbert scheme $(\mathbb{A}^2)^{[n]}$ can be viewed as the first member of the family of the Gieseker moduli spaces $M(r, n)$, corresponding to $r = 1$. The purpose of this section is to generalize the results of Section 2 to the case of higher rank r .

3.1. Correspondences and fixed points for $M(r, n)$. We recall some basics on $M(r, n)$.

Let $M(r, n)$ be the Gieseker framed moduli space of torsion free sheaves on \mathbb{P}^2 of rank r and $c_2 = n$. Its \mathbb{C} -points are the isomorphism classes of pairs $\{(E, \Phi)\}$, where E is a torsion free sheaf on \mathbb{P}^2 of rank r and $c_2(E) = n$ which is locally free in a neighborhood of the line $l_\infty = \{(0 : z_1 : z_2)\} \subset \mathbb{P}^2$, while $\Phi : E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}$ (called a *framing at infinity*).

This space has an alternative quiver description (see [Na1, Ch. 2] for details):

$$M(r, n) = \mathcal{M}(r, n)/GL_n(\mathbb{C}), \quad \mathcal{M}(r, n) = \{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0\}^s,$$

where $B_1, B_2 \in \text{End}(\mathbb{C}^n)$, $i \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $j \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^r)$, the $GL_n(\mathbb{C})$ -action is given by $g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$, while the superscript s symbolizes the stability condition “there is no proper subspace $S \subsetneq \mathbb{C}^n$ which contains $\text{Im } i$ and is B_1, B_2 -invariant”.

Consider a natural action of $\mathbb{T}_r = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$ on $M(r, n)$, where $(\mathbb{C}^*)^2$ acts on \mathbb{P}^2 via $(t_1, t_2)([z_0 : z_1 : z_2]) = [z_0 : t_1 z_1 : t_2 z_2]$, while $(\mathbb{C}^*)^r$ acts by rescaling the framing isomorphism. The set $M(r, n)^{\mathbb{T}_r}$ of \mathbb{T}_r -fixed points in $M(r, n)$ is finite and is in bijection with r -partitions of n , collections of r Young diagrams $(\lambda^1, \dots, \lambda^r)$ satisfying $|\lambda^1| + \dots + |\lambda^r| = n$ (see [NY, Proposition 2.9]). For an r -partition $\bar{\lambda} = (\lambda^1, \dots, \lambda^r) \vdash n$, the corresponding point $\xi_{\bar{\lambda}} \in M(r, n)^{\mathbb{T}_r}$ is given by $E_{\bar{\lambda}} = J_{\lambda^1} \oplus \dots \oplus J_{\lambda^r}$, where Φ is given by a sum of natural inclusions $J_{\lambda^j}|_{l_\infty} \hookrightarrow \mathcal{O}_{l_\infty}$.

Let us recall the *Hecke correspondences*, generalizing the correspondence $P[1]$ from Section 2. Consider $\mathcal{M}(r; n, n+1) \subset \mathcal{M}(r, n) \times \mathcal{M}(r, n+1)$ consisting of pairs of tuples $\{(B_1^{(k)}, B_2^{(k)}, i^{(k)}, j^{(k)})\}$ for $k = n, n+1$, such that there exists $\xi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ satisfying

$$\xi B_1^{(n+1)} = B_1^{(n)} \xi, \quad \xi B_2^{(n+1)} = B_2^{(n)} \xi, \quad \xi i^{(n+1)} = i^{(n)}, \quad j^{(n+1)} = j^{(n)} \xi.$$

The stability condition implies ξ is surjective. Therefore $S := \text{Ker } \xi \subset \mathbb{C}^{n+1}$ is a 1-dimensional subspace of $\text{Ker } j^{(n+1)}$ invariant with respect to $B_1^{(n+1)}, B_2^{(n+1)}$. This provides an identification of $\mathcal{M}(r; n, n+1)$ with pairs of $(B_1^{(n+1)}, B_2^{(n+1)}, i^{(n+1)}, j^{(n+1)}) \in \mathcal{M}(r, n+1)$ and a 1-dimensional subspace $S \subset \mathbb{C}^{n+1}$ satisfying the above conditions. Define the Hecke correspondence $M(r; n, n+1) \subset M(r, n) \times M(r, n+1) = \mathcal{M}(r, n) \times \mathcal{M}(r, n+1)/GL_n(\mathbb{C}) \times GL_{n+1}(\mathbb{C})$ to be the image of $\mathcal{M}(r; n, n+1)$. The set $M(r; n, n+1)^{\mathbb{T}_r}$ of \mathbb{T}_r -fixed points in $M(r; n, n+1)$ is in bijection with r -partitions $\bar{\lambda} \vdash n, \bar{\mu} \vdash n+1$ such that $\lambda^j \subseteq \mu^j$ for $1 \leq j \leq r$; the corresponding fixed point will be denoted by $\xi_{\bar{\lambda}, \bar{\mu}}$. We refer the reader to [Na3, Section 5.1] for more details.

Let L_r be the *tautological* line bundle on $M(r; n, n+1)$, \mathfrak{F}_r be the *tautological* rank n vector bundle on $M(r, n)$, $\mathbf{p}_r, \mathbf{q}_r$ be the natural projections from $M(r; n, n+1)$ to $M(r, n)$ and $M(r, n+1)$, correspondingly. Our computations are based on the following well-known result:

Proposition 3.1. (a) The variety $M(r; n, n+1)$ is smooth of complex dimension $2rn + r + 1$.
(b) The \mathbb{T}_r -character of the tangent space to $M(r, n)$ at the \mathbb{T}_r -fixed point $\xi_{\bar{\lambda}}$ equals

$$T_{\bar{\lambda}} = \sum_{a,b=1}^r \left(\sum_{\square \in \lambda^a} t_1^{-a_{\lambda^b}(\square)} t_2^{l_{\lambda^a}(\square)+1} \frac{\chi_b}{\chi_a} + \sum_{\square \in \lambda^b} t_1^{a_{\lambda^a}(\square)+1} t_2^{-l_{\lambda^b}(\square)} \frac{\chi_b}{\chi_a} \right).$$

(c) The \mathbb{T}_r -character of the fiber of the normal bundle of $M(r; n, n+1)$ at $\xi_{\bar{\lambda}, \bar{\mu}}$ equals

$$N_{\bar{\lambda}, \bar{\mu}} = -t_1 t_2 + \sum_{a,b=1}^r \left(\sum_{\square \in \lambda^a} t_1^{-a_{\lambda^b}(\square)} t_2^{l_{\mu^a}(\square)+1} \frac{\chi_b}{\chi_a} + \sum_{\square \in \mu^b} t_1^{a_{\mu^a}(\square)+1} t_2^{-l_{\lambda^b}(\square)} \frac{\chi_b}{\chi_a} \right).$$

3.2. Geometric $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -action II. We generalize Theorem 2.1 for a higher rank r .

Let $'M^r$ be the direct sum of equivariant (complexified) K -groups: $'M^r = \bigoplus_n K^{\mathbb{T}_r}(M(r, n))$. It is a module over $K^{\mathbb{T}_r}(\text{pt}) = \mathbb{C}[\mathbb{T}_r] = \mathbb{C}[t_1, t_2, \chi_1, \dots, \chi_r]$. We define

$$M^r := 'M^r \otimes_{K^{\mathbb{T}_r}(\text{pt})} \text{Frac}(K^{\mathbb{T}_r}(\text{pt})) = 'M^r \otimes_{\mathbb{C}[t_1, t_2, \chi_1, \dots, \chi_r]} \mathbb{C}(t_1, t_2, \chi_1, \dots, \chi_r).$$

It has a natural grading: $M^r = \bigoplus_n M_n^r$, $M_n^r = K^{\mathbb{T}_r}(M(r, n)) \otimes_{K^{\mathbb{T}_r}(\text{pt})} \text{Frac}(K^{\mathbb{T}_r}(\text{pt}))$. According to the localization theorem, restriction to the \mathbb{T}_r -fixed point set induces an isomorphism

$$K^{\mathbb{T}_r}(M(r, n)) \otimes_{K^{\mathbb{T}_r}(\text{pt})} \text{Frac}(K^{\mathbb{T}_r}(\text{pt})) \xrightarrow{\sim} K^{\mathbb{T}_r}(M(r, n)^{\mathbb{T}_r}) \otimes_{K^{\mathbb{T}_r}(\text{pt})} \text{Frac}(K^{\mathbb{T}_r}(\text{pt})).$$

The structure sheaves $\{\bar{\lambda}\}$ of the \mathbb{T}_r -fixed points $\xi_{\bar{\lambda}}$ (defined in Section 3.1) form a basis in $\bigoplus_n K^{\mathbb{T}_r}(M(r, n)^{\mathbb{T}_r}) \otimes_{K^{\mathbb{T}_r}(\text{pt})} \text{Frac}(K^{\mathbb{T}_r}(\text{pt}))$. Since embedding of a point $\xi_{\bar{\lambda}}$ into $M(r, |\bar{\lambda}|)$ is a proper morphism, the direct image in the equivariant K -theory is well defined, and we denote by $[\bar{\lambda}] \in M_{|\bar{\lambda}|}^r$ the direct image of the structure sheaf $\{\bar{\lambda}\}$. The set $\{[\bar{\lambda}]\}$ forms a basis of M^r .

Consider the generating series $\mathbf{a}_r(z)$, $\mathbf{c}_r(z) \in M^r(z)$ defined as follows:

$$\mathbf{a}_r(z) := \Lambda_{-1/z}^\bullet(\mathfrak{F}_r) = \sum_{i \geq 0} [\Lambda^i(\mathfrak{F}_r)](-1/z)^i,$$

$$\mathbf{c}_r(z) := \mathbf{a}_r(z t_1) \mathbf{a}_r(z t_2) \mathbf{a}_r(z t_3) \mathbf{a}_r(z t_1^{-1})^{-1} \mathbf{a}_r(z t_2^{-1})^{-1} \mathbf{a}_r(z t_3^{-1})^{-1}.$$

Finally, we define the linear operators e_i, f_i, ψ_j^\pm ($i \in \mathbb{Z}, j \in \mathbb{Z}_+$) on M^r :

$$(4) \quad e_i = \mathbf{q}_{r*}(L_r^{\otimes i} \otimes \mathbf{p}_r^*) : M_n^r \rightarrow M_{n+1}^r,$$

$$(5) \quad f_i = \mathbf{p}_{r*}(L_r^{\otimes(i-r)} \otimes \mathbf{q}_r^*) : M_n^r \rightarrow M_{n-1}^r,$$

$$(6) \quad \psi^\pm(z)|_{M_n^r} = \sum_{r=0}^\infty \psi_r^\pm z^{\mp r} := \left((-1)^r t_1 t_2 \chi_1 \dots \chi_r \prod_{a=1}^r \frac{1 - t_1 t_2 \chi_a z}{1 - \chi_a z} \cdot \mathbf{c}_r(z) \right)^\pm \in M_n^r[[z^{\mp 1}]].$$

Theorem 3.2. *The operators e_i, f_i, ψ_j^\pm , defined in (4)-(6), satisfy the relations (T0)-(T6) with the parameters $q_i = t_i$. This endows M^r with the structure of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representation.*

Let us compute the matrix coefficients of those operators in the fixed point basis.

Lemma 3.3. *Consider the fixed point basis $\{[\bar{\lambda}]\}$ of M^r . Define $\chi_k^{(a)} := t_1^{\lambda_k^a - 1} t_2^{k-1} \chi_a^{-1}$.*

(a) *The only nonzero matrix coefficients of the operators e_p, f_p are as follows:*

$$e_{p[\bar{\lambda} - \square_j^l, \bar{\lambda}]} = \frac{(\chi_j^{(l)})^p}{1 - t_1^{-1} t_2^{-1}} \cdot \prod_{a=1}^r \prod_{k=1}^\infty \frac{1 - t_1 t_2 \chi_k^{(a)} / \chi_j^{(l)}}{1 - t_1 \chi_k^{(a)} / \chi_j^{(l)}},$$

$$f_{p[\bar{\lambda} + \square_j^l, \bar{\lambda}]} = \frac{(t_1 \chi_j^{(l)})^{p-r}}{1 - t_1^{-1} t_2^{-1}} \cdot \prod_{a=1}^r \prod_{k=1}^\infty \frac{1 - t_1 t_2 \chi_j^{(l)} / \chi_k^{(a)}}{1 - t_1 \chi_j^{(l)} / \chi_k^{(a)}},$$

where $\bar{\lambda} \pm \square_j^l$ denotes the r -partition obtained from $\bar{\lambda}$ by adding/erasing a box in j -th row of λ^l .

(b) *The eigenvalue of $\psi^\pm(z)$ on $[\bar{\lambda}]$ equals*

$$\left((-1)^r t_1^{r+1} t_2^{r+1} \chi_1 \dots \chi_r \prod_{a=1}^r \frac{1 - t_3 \chi_a^{-1} / z}{1 - \chi_a^{-1} / z} \prod_{a=1}^r \prod_{\square \in \lambda^a} \frac{(1 - t_1^{-1} \chi(\square) / z)(1 - t_2^{-1} \chi(\square) / z)(1 - t_3^{-1} \chi(\square) / z)}{(1 - t_1 \chi(\square) / z)(1 - t_2 \chi(\square) / z)(1 - t_3 \chi(\square) / z)} \right)^\pm$$

where $\chi(\square_{i,j}^a) = t_1^{i-1} t_2^{j-1} \chi_a^{-1}$ for a box $\square_{i,j}^a$ staying in the j -th row and i -th column of λ^a .

This lemma allows to prove Theorem 3.2 just by a straightforward verification of the relations (T0)-(T6) in the fixed point basis. The only nontrivial relation (T3) can be verified analogously to the case of $(\mathbb{A}^2)^{[n]}$. We refer the reader to Appendix B for more details.

3.3. Geometric $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -action II. We generalize Theorem 2.2 for a higher rank r .

Let $'V^r$ be the direct sum of equivariant (complexified) cohomology: $'V^r = \bigoplus_n H_{\mathbb{T}_r}^\bullet(M(r, n))$. It is a module over $H_{\mathbb{T}_r}^\bullet(\text{pt}) = \mathbb{C}[\mathfrak{t}_r] = \mathbb{C}[s_1, s_2, x_1, \dots, x_r]$, where $\mathfrak{t}_r = \text{Lie}(\mathbb{T}_r)$. We define

$$V^r := 'V^r \otimes_{H_{\mathbb{T}_r}^\bullet(\text{pt})} \text{Frac}(H_{\mathbb{T}_r}^\bullet(\text{pt})) = 'V^r \otimes_{\mathbb{C}[s_1, s_2, x_1, \dots, x_r]} \mathbb{C}(s_1, s_2, x_1, \dots, x_r).$$

It has a natural grading: $V^r = \bigoplus_n V_n^r$, $V_n^r = H_{\mathbb{T}_r}^\bullet(M(r, n)) \otimes_{H_{\mathbb{T}_r}^\bullet(\text{pt})} \text{Frac}(H_{\mathbb{T}_r}^\bullet(\text{pt}))$. According to the localization theorem, restriction to the \mathbb{T}_r -fixed point set induces an isomorphism

$$H_{\mathbb{T}_r}^\bullet(M(r, n)) \otimes_{H_{\mathbb{T}_r}^\bullet(\text{pt})} \text{Frac}(H_{\mathbb{T}_r}^\bullet(\text{pt})) \xrightarrow{\sim} H_{\mathbb{T}_r}^\bullet(M(r, n)^{\mathbb{T}_r}) \otimes_{H_{\mathbb{T}_r}^\bullet(\text{pt})} \text{Frac}(H_{\mathbb{T}_r}^\bullet(\text{pt})).$$

The fundamental cycles $[\bar{\lambda}]$ of the \mathbb{T}_r -fixed points $\xi_{\bar{\lambda}}$ form a basis in $\bigoplus_n H_{\mathbb{T}_r}^\bullet(M(r, n)^{\mathbb{T}_r}) \otimes_{H_{\mathbb{T}_r}^\bullet(\text{pt})} \text{Frac}(H_{\mathbb{T}_r}^\bullet(\text{pt}))$. Since embedding of a point $\xi_{\bar{\lambda}}$ into $M(r, |\bar{\lambda}|)$ is a proper morphism, the direct image in the equivariant cohomology is well defined, and we will denote by $[\bar{\lambda}] \in V_{|\bar{\lambda}|}^r$ the direct image of the fundamental cycle of the point $\xi_{\bar{\lambda}}$. The set $\{[\bar{\lambda}]\}$ forms a basis of V^r .

We introduce the generating series $\mathbf{C}_r(z) \in V^r[[z^{-1}]]$ as follows:

$$\mathbf{C}_r(z) := \left(\frac{\text{ch}(\mathfrak{F}_r t_1^{-1}, -z^{-1}) \text{ch}(\mathfrak{F}_r t_2^{-1}, -z^{-1}) \text{ch}(\mathfrak{F}_r t_3^{-1}, -z^{-1})}{\text{ch}(\mathfrak{F}_r t_1, -z^{-1}) \text{ch}(\mathfrak{F}_r t_2, -z^{-1}) \text{ch}(\mathfrak{F}_r t_3, -z^{-1})} \right)^+.$$

Finally, we define the linear operators e_j, f_j, ψ_j ($j \in \mathbb{Z}_+$) on V^r :

$$(4') \quad e_j = \mathbf{q}_{r*}(c_1(L_r)^j \cdot \mathbf{p}_r^*) : V_n^r \rightarrow V_{n+1}^r,$$

$$(5') \quad f_j = (-1)^{r-1} \mathbf{p}_{r*}(c_1(L_r)^j \cdot \mathbf{q}_r^*) : V_n^r \rightarrow V_{n-1}^r,$$

$$(6') \quad \psi(z)|_{V_n^r} = 1 + s_1 s_2 s_3 \sum_{r=0}^{\infty} \psi_r z^{-r-1} := \left(\prod_{a=1}^r \frac{1 + \frac{x_a - s_3}{z}}{1 + \frac{x_a}{z}} \cdot \mathbf{C}_r(z) \right)^+ \in V_n^r[[z^{-1}]].$$

Theorem 3.4. *The operators e_j, f_j, ψ_j , defined in (4')-(6'), satisfy the relations (Y0)-(Y6) with the parameters $h_i = s_i$. This endows V^r with the structure of $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representation.*

Let us compute the matrix coefficients of those operators in the fixed point basis.

Lemma 3.5. *Consider the fixed point basis $\{[\bar{\lambda}]\}$ of V^r . Define $x_k^{(a)} := (\lambda_k^a - 1)s_1 + (k-1)s_2 - x_a$. (a) The only nonzero matrix coefficients of the operators e_p, f_p are as follows:*

$$e_{p[\bar{\lambda} - \square_j^l, \bar{\lambda}]} = \frac{(x_j^{(l)})^p}{s_1 + s_2} \cdot \prod_{a=1}^r \prod_{k=1}^{\infty} \frac{s_1 + s_2 + x_k^{(a)} - x_j^{(l)}}{s_1 + x_k^{(a)} - x_j^{(l)}},$$

$$f_{p[\bar{\lambda} + \square_j^l, \bar{\lambda}]} = (-1)^{r-1} \frac{(s_1 + x_j^{(l)})^p}{s_1 + s_2} \cdot \prod_{a=1}^r \prod_{k=1}^{\infty} \frac{s_1 + s_2 + x_j^{(l)} - x_k^{(a)}}{s_1 + x_j^{(l)} - x_k^{(a)}}.$$

(b) *The eigenvalue of $\psi(z)$ on $[\bar{\lambda}]$ equals*

$$\left(\prod_{a=1}^r \frac{1 + \frac{x_a - s_3}{z}}{1 + \frac{x_a}{z}} \cdot \prod_{a=1}^r \prod_{\square \in \lambda^a} \frac{(1 - \frac{\chi(\square) - s_1}{z})(1 - \frac{\chi(\square) - s_2}{z})(1 - \frac{\chi(\square) - s_3}{z})}{(1 - \frac{\chi(\square) + s_1}{z})(1 - \frac{\chi(\square) + s_2}{z})(1 - \frac{\chi(\square) + s_3}{z})} \right)^+,$$

where $\chi(\square_{i,j}^a) = (i-1)s_1 + (j-1)s_2 - x_a$.

This lemma allows to prove Theorem 3.4 just by a straightforward verification of the relations (Y0)-(Y6) in the fixed point basis. The only nontrivial relation is actually (Y3). Its proof is based on the statement analogous to Lemma 2.4; see Appendix B for more details.

Corollary 3.6. $\psi(z)|_{\bar{\lambda}} = 1 - \frac{rs_3}{z} + \frac{s_3 \sum x_j + \binom{r}{2} s_3^2}{z^2} + \frac{2\sigma_3 |\bar{\lambda}| - s_3 \sum x_j^2 - (r-1)s_3^2 \sum x_j - \binom{r}{3} s_3^3}{z^3} + o(z^{-3})$.

4. ON SOME REPRESENTATIONS OF $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ AND $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$

In this section, we recall several families of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representations from [FFJMM1, FFJMM2] and establish their analogues for the case of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$. This should be viewed as an analogy between the representation theory of $U_q(L\mathfrak{g})$ and $Y_h(\mathfrak{g})$.

4.1. Vector representations. We start from the simplest representations $V(u)$ and $V^a(u)$.

The main building block of all constructions is the family of *vector representations* of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$, whose basis is parametrized by \mathbb{Z} (see [FFJMM1, Proposition 3.1]).

Proposition 4.1 (Vector representation of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}^*$, let $V(u)$ be a \mathbb{C} -vector space with the basis $\{[u]_j\}_{j \in \mathbb{Z}}$. The following formulas define $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -action on $V(u)$:*

$$\begin{aligned} e(z)[u]_i &= (1 - q_1)^{-1} \delta(q_1^i u/z) \cdot [u]_{i+1}, \\ f(z)[u]_i &= (q_1^{-1} - 1)^{-1} \delta(q_1^{i-1} u/z) \cdot [u]_{i-1}, \\ \psi^\pm(z)[u]_i &= \left(\frac{(z - q_1^i q_2 u)(z - q_1^i q_3 u)}{(z - q_1^i u)(z - q_1^{i-1} u)} \right)^\pm \cdot [u]_i. \end{aligned}$$

Analogously to that, we define a family of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ *vector representations*:

Proposition 4.2 (Vector representation of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}$, let ${}^a V(u)$ be a \mathbb{C} -vector space with the basis $\{[u]_j\}_{j \in \mathbb{Z}}$. The following formulas define $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -action on ${}^a V(u)$:*

$$\begin{aligned} e(z)[u]_i &= \frac{1}{h_1 z} \delta^+((ih_1 + u)/z) [u]_{i+1} = \left(\frac{1}{h_1(z - u - ih_1)} \right)^+ \cdot [u]_{i+1}, \\ f(z)[u]_i &= -\frac{1}{h_1 z} \delta^+(((i-1)h_1 + u)/z) [u]_{i-1} = \left(\frac{-1}{h_1(z - u - (i-1)h_1)} \right)^+ \cdot [u]_{i-1}, \\ \psi(z)[u]_i &= \left(\frac{(z - (ih_1 + h_2 + u))(z - (ih_1 + h_3 + u))}{(z - (ih_1 + u))(z - ((i-1)h_1 + u))} \right)^+ \cdot [u]_i, \end{aligned}$$

where $\delta^+(w) := 1 + w + w^2 + \dots = (\frac{1}{1-w})^+$.

4.2. Fock representations. Next, we introduce a family of Fock modules $F(u)$ and ${}^a F(u)$.

A more interesting family of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representations, whose basis is parametrized by all Young diagrams $\{\lambda\}$, was established in [FFJMM1, Theorem 4.3, Corollary 4.4].

Proposition 4.3 (Fock representation of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}^*$, let $F(u)$ be a \mathbb{C} -vector space with the basis $\{|\lambda\rangle\}$. The following formulas define $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -action on $F(u)$:*

$$\begin{aligned} e(z)|\lambda\rangle &= \sum_{i \geq 1} \prod_{j=1}^{i-1} \frac{(1 - q_1^{\lambda_i - \lambda_j} q_2^{i-j-1})(1 - q_1^{\lambda_i - \lambda_j + 1} q_2^{i-j+1})}{(1 - q_1^{\lambda_i - \lambda_j} q_2^{i-j})(1 - q_1^{\lambda_i - \lambda_j + 1} q_2^{i-j})} \cdot \frac{\delta(q_1^{\lambda_i} q_2^{i-1} u/z)}{1 - q_1} \cdot |\lambda + i\rangle, \\ f(z)|\lambda\rangle &= \sum_{i \geq 1} \frac{1 - q_1^{\lambda_{i+1} - \lambda_i}}{1 - q_1^{\lambda_{i+1} - \lambda_i + 1} q_2} \prod_{j=i+1}^{\infty} \frac{(1 - q_1^{\lambda_j - \lambda_{i+1}} q_2^{j-i+1})(1 - q_1^{\lambda_{j+1} - \lambda_i} q_2^{j-i})}{(1 - q_1^{\lambda_{j+1} - \lambda_{i+1} + 1} q_2^{j-i+1})(1 - q_1^{\lambda_j - \lambda_i} q_2^{j-i})} \cdot \frac{\delta(q_1^{\lambda_{i+1}} q_2^{i-1} u/z)}{q_1^{-1} - 1} \cdot |\lambda - i\rangle, \\ \psi^\pm(z)|\lambda\rangle &= \left(\frac{z - q_1^{\lambda_1 - 1} q_2^{-1} u}{z - q_1^{\lambda_1} u} \prod_{i=1}^{\infty} \frac{(z - q_1^{\lambda_i} q_2^i u)(z - q_1^{\lambda_{i+1} - 1} q_2^{i-1} u)}{(z - q_1^{\lambda_{i+1}} q_2^i u)(z - q_1^{\lambda_i - 1} q_2^{i-1} u)} \right)^\pm \cdot |\lambda\rangle. \end{aligned}$$

Remark 4.1. The Fock module $F(u)$ was originally constructed from $V(u)$ by using the semi-infinite wedge construction and the coproduct structure on $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ defined by:

$$\Delta(e(z)) = e(z) \otimes 1 + \psi^-(z) \otimes e(z), \quad \Delta(f(z)) = f(z) \otimes \psi^+(z) + 1 \otimes f(z), \quad \Delta(\psi^\pm(z)) = \psi^\pm(z) \otimes \psi^\pm(z).$$

Let us also recall the relation between $F(u)$ and the module M from Theorem 2.1.

Remark 4.2. (a) According to [FFJMM1, Corollary 4.5], there exist constants $\{c_\lambda\}$ such that the map $[\lambda] \mapsto c_\lambda |\lambda\rangle$ establishes an isomorphism $M \xrightarrow{\sim} F(1)$ of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representations.
 (b) Let ϕ_u be the *shift automorphism* of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ defined on the generators by

$$e_r \mapsto u^r \cdot e_r, \quad f_r \mapsto u^r \cdot f_r, \quad \psi_j^\pm \mapsto u^{\pm j} \cdot \psi_j^\pm, \quad r \in \mathbb{Z}, j \in \mathbb{Z}_+.$$

Then the modules $F(u)$ and $V(u)$ are obtained from $F(1)$ and $V(1)$ via a twist by ϕ_u .

This construction also has an analogue in the $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -case.

Proposition 4.4 (Fock representation of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}$, let ${}^aF(u)$ be a \mathbb{C} -vector space with the basis $\{|\lambda\rangle\}$. The following formulas define $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -action on ${}^aF(u)$:*

$$\begin{aligned} e(z)|\lambda\rangle &= \frac{1}{h_1 z} \sum_{i \geq 1} \prod_{j=1}^{i-1} \frac{((\lambda_i - \lambda_j)h_1 + (i-j-1)h_2)((\lambda_i - \lambda_j + 1)h_1 + (i-j+1)h_2)}{((\lambda_i - \lambda_j)h_1 + (i-j)h_2)((\lambda_i - \lambda_j + 1)h_1 + (i-j)h_2)} \times \\ &\quad \delta^+ \left(\frac{\lambda_i h_1 + (i-1)h_2 + u}{z} \right) \cdot |\lambda + i\rangle, \\ f(z)|\lambda\rangle &= -\frac{1}{h_1 z} \sum_{i \geq 1} \prod_{j=i+1}^{\infty} \frac{((\lambda_j - \lambda_i + 1)h_1 + (j-i+1)h_2)((\lambda_{j+1} - \lambda_i)h_1 + (j-i)h_2)}{((\lambda_{j+1} - \lambda_i + 1)h_1 + (j-i+1)h_2)((\lambda_j - \lambda_i)h_1 + (j-i)h_2)} \times \\ &\quad \frac{(\lambda_{i+1} - \lambda_i)h_1}{(\lambda_{i+1} - \lambda_i + 1)h_1 + h_2} \delta^+ \left(\frac{(\lambda_i - 1)h_1 + (i-1)h_2 + u}{z} \right) \cdot |\lambda - i\rangle, \\ \psi(z)|\lambda\rangle &= \left(\prod_{i=1}^{\infty} \frac{(z - (\lambda_i h_1 + i h_2 + u))(z - ((\lambda_{i+1} - 1)h_1 + (i-1)h_2 + u))}{(z - (\lambda_{i+1} h_1 + i h_2 + u))(z - ((\lambda_i - 1)h_1 + (i-1)h_2 + u))} \right)^+ \cdot \\ &\quad \left(\frac{z - ((\lambda_1 - 1)h_1 - h_2 + u)}{z - (\lambda_1 h_1 + u)} \right)^+ \cdot |\lambda\rangle. \end{aligned}$$

The proof of this proposition follows from the following lemma:

Lemma 4.5. (a) *For $u \in \mathbb{C}$, there exists the shift automorphism ϕ_u^a of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ such that*

$$\phi_u^a : e(z) \mapsto e(z - u), \quad f(z) \mapsto f(z - u), \quad \psi(z) \mapsto \psi(z - u).$$

(b) *The Fock representation ${}^aF(u)$ is obtained from ${}^aF(0)$ via a twist by ϕ_u^a .*

(c) *There exist constants $\{c_\lambda^a\}$ such that the map $[\lambda] \mapsto c_\lambda^a |\lambda\rangle$ establishes an isomorphism $V \xrightarrow{\sim} {}^aF(0)$ of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representations, where V is the representation from Theorem 2.2.*

Proof.

Parts (a) and (b) are straightforward.

We define c_λ^a by the following formula:

$$c_\lambda^a = \prod_{i \geq 1} \prod_{p=0}^{\lambda_i - 1} (-p h_1 + h_2) \cdot \prod_{i \geq 2} \prod_{j=1}^{i-1} \prod_{p=1}^{\lambda_i} \frac{(p - \lambda_j)h_1 + (i-j)h_2}{(p - \lambda_j)h_1 + (i-j+1)h_2}.$$

It is a routine verification to check that the map $[\lambda] \mapsto c_\lambda^a |\lambda\rangle$ intertwines the formulas for the matrix coefficients of e_j, f_j, ψ_j from Lemma 2.3 and Proposition 4.4. \square

Definition 4.1. We say that a representation U of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ has a central charge (c_0, c_1) ($c_i \in \mathbb{C}$) if central elements ψ_i act on U as multiplications by c_i for $i = 0, 1$.

Thus ${}^aV(u)$ has a central charge $(0, \frac{1}{h_1})$, while ${}^aF(u)$ has a central charge $(-\frac{1}{h_1 h_2}, -\frac{u}{h_1 h_2})$.

4.3. The tensor product of Fock modules $F(u)$. In this section, we express the representation M^r from Section 3 as the appropriate tensor product of Fock modules $F(u)$.

Let Δ be the *formal* comultiplication on $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ from Remark 4.1. This is not a comultiplication in the usual sense, since $\Delta(e_i)$ and $\Delta(f_i)$ contain infinite sums. However, for all modules of our concern, these formulas make sense. Recall the $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representation M^r , constructed in Theorem 3.2. Let κ be the automorphism of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$, defined on the generators by $\kappa(e_i) = e_i$, $\kappa(f_i) = T^{-1}f_i$, $\kappa(\psi_i^\pm) = T^{-1}\psi_i^\pm$, where $T = (t_1 t_2)^{r+1} \chi_1 \cdots \chi_r$. Let \bar{M}^r be the $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representation, obtained from M^r via a twist by κ .

Theorem 4.6. *There exists a unique collection of constants $\{c_{\bar{\lambda}}\}$ from $\mathbb{C}(t_1, t_2, \chi_1, \dots, \chi_r)$ with $c_{\emptyset} = 1$ such that the map $[\bar{\lambda}] = [(\lambda^1, \dots, \lambda^r)] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ establishes an isomorphism $\bar{M}^r \xrightarrow{\sim} F(\chi_1) \otimes \cdots \otimes F(\chi_r)$ of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representations.*

Let us first comment on the tensor product $F(\chi_1) \otimes F(\chi_2)$ (the case $r > 2$ is completely analogous). In order to make sense of the *formal* coproduct in this setting, note that

$$e(z)|\lambda\rangle = \sum_{\square} a_{\lambda, \square} \delta\left(\frac{\chi(\square)}{z}\right) |\lambda + \square\rangle, \quad f(z)|\lambda\rangle = \sum_{\square} b_{\lambda, \square} \delta\left(\frac{\chi(\square)}{z}\right) |\lambda - \square\rangle,$$

where $a_{\lambda, \square}, b_{\lambda, \square} \in \mathbb{C}(t_1, t_2, \chi_1, \dots, \chi_r)$, the first sum is over $\square \notin \lambda$ such that $\lambda + \square$ is a Young diagram, while the second sum is over $\square \in \lambda$ such that $\lambda - \square$ is a Young diagram.

According to the coproduct formula, we have

$$\Delta(e(z))(|\lambda^1\rangle \otimes |\lambda^2\rangle) = e(z)(|\lambda^1\rangle) \otimes |\lambda^2\rangle + \psi^-(z)(|\lambda^1\rangle) \otimes e(z)(|\lambda^2\rangle).$$

The first summand is well defined. To make sense of the second summand we use the formula

$$(*) \quad g(z)\delta(a/z) = g(a)\delta(a/z).$$

Recall that $\psi^\pm(z)(|\lambda\rangle) = \gamma_\lambda(z)^\pm \cdot |\lambda\rangle$, where $\gamma_\lambda(z)$ is a rational function in z depending on λ . Combining this with (*), we rewrite

$$\psi^-(z)(|\lambda^1\rangle) \otimes e(z)(|\lambda^2\rangle) = \sum_{\square} a_{\lambda^2, \square} \gamma_{\lambda^1}(\chi(\square)) \delta\left(\frac{\chi(\square)}{z}\right) \cdot |\lambda^1\rangle \otimes |\lambda^2 + \square\rangle.$$

Analogously we make sense of the formula for the action of f_i on $F(\chi_1) \otimes F(\chi_2)$. Finally, the formula $\Delta(\psi^\pm(z)) = \psi^\pm(z) \otimes \psi^\pm(z)$ provides a well-defined action of ψ_i^\pm on $F(\chi_1) \otimes F(\chi_2)$.

Proof of Theorem 4.6.

Due to Remark 4.2, we identify $F(\chi_j) \simeq M^{\phi_{\chi_j}}$, the twist of M by the shift automorphism ϕ_{χ_j} . For any r -partition $\bar{\lambda} = (\lambda^1, \dots, \lambda^r)$, Lemma 3.3(b) implies that the eigenvalue of $\psi^\pm(z)$ on $[\bar{\lambda}] \in \bar{M}^r$ equals the eigenvalue of $\psi^\pm(z)$ on $|\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle \in F(\chi_1) \otimes \cdots \otimes F(\chi_r)$. Hence, for any constants $c_{\bar{\lambda}}$ the map $[\bar{\lambda}] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ intertwines actions of $\{\psi_j^\pm\}_{j \geq 0}$.

Consider constants $c_{\bar{\lambda}}$ defined by $c_{\emptyset} = 1$ and $c_{\bar{\lambda} + \square_j^l} / c_{\bar{\lambda}} = d_{\bar{\lambda}, \square_j^l}$, where

$$(7) \quad d_{\bar{\lambda}, \square_j^l} := (-1)^{l-1} t_1^{-1} t_2^{-1} \cdot \prod_{a=l+1}^r \prod_{k=1}^{\infty} \frac{\chi_j^{(l)} - \chi_k^{(a)}}{\chi_j^{(l)} - t_2 \chi_k^{(a)}} \cdot \prod_{a=1}^{l-1} \prod_{k=1}^{\infty} \frac{\chi_j^{(l)} - t_1^{-1} t_2^{-1} \chi_k^{(a)}}{\chi_j^{(l)} - t_1^{-1} \chi_k^{(a)}}.$$

Here $\chi_p^{(m)} = t_1^{\lambda_p^{(m)} - 1} t_2^{p-1} \chi_m^{-1}$ and $\bar{\lambda} + \square_j^k$ denotes the r -partition obtained from $\bar{\lambda}$ by adding a box to the j -th row of λ^k . Note that $\chi_{p+1}^{(m)} = t_2 \chi_p^{(m)}$ for $p \geq |n|$ and so the infinite products of (7) are actually finite. It is straightforward to check that $c_{\bar{\lambda}}$ are well-defined, that is, $d_{\bar{\lambda}, \square_j^k}$ satisfy $d_{\bar{\lambda} + \square_j^k, \square_a^l} d_{\bar{\lambda}, \square_j^k} = d_{\bar{\lambda} + \square_a^l, \square_j^k} d_{\bar{\lambda}, \square_a^l}$. Using Lemma 3.3(a), it is straightforward to check that the map $[\bar{\lambda}] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ intertwines actions of e_i and f_i as well. The result follows. \square

4.4. The tensor product of Fock modules ${}^aF(u)$. In this section, we express the representation V^r from Section 3 as the appropriate tensor product of Fock modules ${}^aF(u)$. To formulate the result, we need to define the tensor product $W_1 \otimes W_2$ of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representations W_i .

The action of $\psi(z)$ on $W_1 \otimes W_2$ is defined via the comultiplication $\Delta(\psi(z)) = \psi(z) \otimes \psi(z)$, that is, $\psi(z)(w^1 \otimes w^2) = \psi(z)(w^1) \otimes \psi(z)(w^2) \forall w^1 \in W_1, w^2 \in W_2$. To define the action of $e(z), f(z)$ on $W_1 \otimes W_2$, we should restrict to a particular class of representations. A $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representation W is called *admissible* if there exists a basis $\{w_\alpha\}_{\alpha \in I}$ of W such that

$\circ e(z)(w_\alpha) = \sum_{\alpha' \in I} \frac{c_{\alpha, \alpha'}}{z} \delta^+(\lambda_{\alpha, \alpha'}/z) w_{\alpha'}, f(z)(w_\alpha) = \sum_{\alpha'' \in I} \frac{d_{\alpha, \alpha''}}{z} \delta^+(\lambda_{\alpha'', \alpha}/z) w_{\alpha''}$ for some $c_{\alpha, \alpha'}, d_{\alpha, \alpha''}, \lambda_{\alpha, \alpha'} \in \mathbb{C}$. For each α , both sums have only finite number of nonzero summands.

$\circ \psi(z)(w_\alpha) = \gamma_W(\alpha, z) \cdot w_\alpha$ for a rational function $\gamma_W(\alpha, \bullet)$ defined by

$$\gamma_W(\alpha, \bullet) = 1 + \sigma_3 \sum_{\alpha'' \in I} \frac{d_{\alpha, \alpha''} c_{\alpha'', \alpha}}{z - \lambda_{\alpha'', \alpha}} - \sigma_3 \sum_{\alpha' \in I} \frac{c_{\alpha, \alpha'} d_{\alpha', \alpha}}{z - \lambda_{\alpha, \alpha'}}.$$

Example 4.1. The modules ${}^aV(u)$ and ${}^aF(u)$ are admissible.

Let W_1, W_2 be admissible $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representations with the corresponding bases $\{w_\alpha^1\}_{\alpha \in I}$ and $\{w_\beta^2\}_{\beta \in J}$. Consider the operator series $e(z), f(z)$ on $W_1 \otimes W_2$ defined by

$$\begin{aligned} e(z)(w_\alpha^1 \otimes w_\beta^2) &:= \sum_{\alpha' \in I} \frac{c_{\alpha, \alpha'}}{z} \delta^+(\lambda_{\alpha, \alpha'}/z) w_{\alpha'}^1 \otimes w_\beta^2 + \sum_{\beta' \in J} \frac{c_{\beta, \beta'}^2 \gamma_{W_1}(\alpha, \lambda_{\beta, \beta'}^2)}{z} \delta^+(\lambda_{\beta, \beta'}/z) w_\alpha^1 \otimes w_{\beta'}^2. \\ f(z)(w_\alpha^1 \otimes w_\beta^2) &:= \sum_{\beta'' \in J} \frac{d_{\beta, \beta''}^2}{z} \delta^+(\lambda_{\beta'', \beta}/z) w_\alpha^1 \otimes w_{\beta''}^2 + \sum_{\alpha'' \in I} \frac{d_{\alpha, \alpha''}^1 \gamma_{W_2}(\beta, \lambda_{\alpha'', \alpha}^1)}{z} \delta^+(\lambda_{\alpha'', \alpha}/z) w_{\alpha''}^1 \otimes w_\beta^2. \end{aligned}$$

Remark 4.3. Those formulas are well-defined only if for any β' such that $c_{\beta, \beta'}^2 \neq 0$, the function $\gamma_{W_1}(\alpha, z)$ is regular at $z = \lambda_{\beta, \beta'}^2$ for any $\alpha \in I$, and similarly for the summand with $\gamma_{W_2}(\beta, z)$.

We can depict those by $\Delta(e(z)) = e(z) \otimes 1 + \psi(\bullet) \otimes e(z)$, $\Delta(f(z)) = f(z) \otimes \psi(\bullet) + 1 \otimes f(z)$, where $\psi(\bullet)$ indicates that we plug in the argument of the corresponding δ^+ -function.

The following is straightforward:

Lemma 4.7. *If W_1 and W_2 are admissible $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representations and the assumptions of Remark 4.3 hold, then the above formulas define an action of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ on $W_1 \otimes W_2$.*

More importantly, it might be possible to define an action of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ on a submodule or a factor-module of $W_1 \otimes W_2$, even when the assumptions of Remark 4.3 fail.

Lemma 4.8. *Let S be a subset of $I \times J$ such that $e(z)(w_\alpha^1 \otimes w_\beta^2), f(z)(w_\alpha^1 \otimes w_\beta^2)$ are well-defined (in the sense of Remark 4.3) for any $(\alpha, \beta) \in S$ and satisfy one of the following conditions:*

- (a) *For any $(\alpha, \beta) \in S, (\alpha', \beta') \notin S$, $w_\alpha^1 \otimes w_{\beta'}^2$ doesn't appear in $e(z)(w_\alpha^1 \otimes w_\beta^2), f(z)(w_\alpha^1 \otimes w_\beta^2)$.*
- (b) *For any $(\alpha, \beta) \in S, (\alpha', \beta') \notin S$, $w_\alpha^1 \otimes w_\beta^2$ doesn't appear in $e(z)(w_{\alpha'}^1 \otimes w_{\beta'}^2), f(z)(w_{\alpha'}^1 \otimes w_{\beta'}^2)$.*

Then the above formulas define $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -action on the space with a basis $\{w_\alpha^1 \otimes w_\beta^2\}_{(\alpha, \beta) \in S}$.

Now we are ready to state the main result of this subsection:

Theorem 4.9. *There exists a unique collection of constants $\{c_\lambda^a\}$ from $\mathbb{C}(s_1, s_2, x_1, \dots, x_r)$ with $c_\emptyset^a = 1$ such that the map $[\bar{\lambda}] = [(\lambda^1, \dots, \lambda^r)] \mapsto c_\lambda^a \cdot |\lambda^1\rangle \otimes \dots \otimes |\lambda^r\rangle$ establishes an isomorphism $V^r \xrightarrow{\sim} {}^aF(x_1) \otimes \dots \otimes {}^aF(x_r)$ of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representations.*

Remark 4.4. As $V \simeq V^1$, we have $V^r \simeq V^1(x_1) \otimes \dots \otimes V^1(x_r)$. In other words, the representation of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ on the sum of equivariant cohomology groups of $M(r, n)$ is a tensor product of r copies of such representations for $(\mathbb{A}^2)^{[m]}$.

4.5. Other series of representations. We discuss other known series of representations.

We recall some other series of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -representations from [FFJMM1, FFJMM2]. All of them admit a straightforward modification to the $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -case. Those have the same bases, while the matrix coefficients of $e(z), f(z), \psi(z)$ in these bases are modified as follows:

$$1 - q_1^i q_2^j q_3^k u/z \rightsquigarrow ih_1 + jh_2 + kh_3 + u - z, \quad \delta(q_1^i q_2^j q_3^k u/z) \rightsquigarrow \pm \frac{1}{z} \delta^+((ih_1 + jh_2 + kh_3 + u)/z),$$

where the latter sign is $+$ for $e(z)$ and $-$ for $f(z)$.

- *Representation $W^N(u)$.*

Consider the tensor product $V^N(u) := V(u) \otimes V(uq_3^{-1}) \otimes V(uq_3^{-2}) \otimes \cdots \otimes V(uq_3^{1-N})$. Define $\mathcal{P}^N := \{\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N | \lambda_1 \geq \cdots \geq \lambda_N\}$, $\mathcal{P}^{N,+} := \{\lambda \in \mathcal{P}^N | \lambda_N \geq 0\}$. Let $W^N(u) \subset V^N(u)$ be the subspace spanned by $[u]_\lambda := [u]_{\lambda_1} \otimes [uq_3^{-1}]_{\lambda_2-1} \otimes \cdots \otimes [uq_3^{1-N}]_{\lambda_N-N+1}$ for $\lambda \in \mathcal{P}^N$.

According to [FFJMM1, Lemma 3.7], $W^N(u)$ is a $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -submodule of $V^N(u)$. The subspace $W^{N,+}(u) \subset W^N(u)$ corresponding to $\mathcal{P}^{N,+}$ is not a submodule. However, its limit as $N \rightarrow \infty$ is well-defined (after a renormalization) and coincides with the Fock module $F(u)$.

- *Representation $G_{\mathbf{a}}^{k,r}$.*

Let q_1, q_2 be in the (r, k) -resonance condition: $q_1^a q_2^b = 1$ iff $a = (1-r)c, b = (k+1)c$ for some $c \in \mathbb{Z}$ (assume $k \geq 1, r \geq 2$). In this case the action of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ on $W^N(u)$ is ill-defined. Consider the set of (k, r) -admissible partitions $S^{k,r,N} := \{\lambda \in \mathcal{P}^N | \lambda_i - \lambda_{i+k} \geq r \ \forall i \leq N-k\}$.

Let $W^{k,r,N}(u)$ be the subspace of $W^N(u)$, corresponding to the subset $S^{k,r,N} \subset \mathcal{P}^N$. According to [FFJMM1, Lemma 6.2], the comultiplication rule makes $W^{k,r,N}(u)$ into a $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -module. We think of it as “a submodule of $W^N(u)$ or even $V^N(u)$ ” even though none of them has a $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -module structure (we use Lemma 4.8 here).

Moreover, one can define an action of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ on the corresponding limit of $W^{k,r,N}(u)$ as $N \rightarrow \infty$. Let us fix a sequence of non-negative integers $\mathbf{a} = (a_1, \dots, a_k)$ satisfying $\sum_{i=1}^k a_i = r$. Define $\mathcal{P}_{\mathbf{a}}^{k,r} := \{(\lambda_1 \geq \lambda_2 \geq \cdots) | \lambda_j - \lambda_{j+k} \geq r \ \forall j \geq 1, \lambda_j = \lambda_j^0 \ \forall j \gg 0\}$, where we set $\lambda_{\mu k + i + 1}^0 := -\mu r - \sum_{j=1}^i a_j$ for $0 \leq i \leq k-1$. The above limit construction provides an action of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ on the space $G_{\mathbf{a}}^{k,r}$ parametrized by $\lambda \in \mathcal{P}_{\mathbf{a}}^{k,r}$, see [FFJMM1, Theorem 6.5]

- *Representation $\mathcal{M}_{\mathbf{a}, \mathbf{b}}(u)$.*

Let us consider the tensor product of Fock representations. If $q_1, q_2, u_1, \dots, u_n$ are generic ($q_1^a q_2^b u_1^{c_1} \cdots u_n^{c_n} = 1$ iff $a = b = c_1 = \dots = c_n = 0$), then the tensor product $F(u_1) \otimes \cdots \otimes F(u_n)$ is well-defined. Consider the resonance case $u_i = u_{i+1} q_1^{a_i+1} q_2^{b_i+1}$ for some $a_i, b_i \geq 0, 1 \leq i \leq n-1$.

Let $\mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \subset F(u_1) \otimes \cdots \otimes F(u_n)$ be the subspace spanned by $|\lambda^1, \dots, \lambda^n\rangle := [u_1]_{\lambda^1} \otimes \cdots \otimes [u_n]_{\lambda^n}$, where Young diagrams $\lambda^1, \dots, \lambda^n$ satisfy $\lambda_s^i \geq \lambda_{s+b_i}^{i+1} - a_i$ for $i \leq n-1, s \geq 1$. According to [FFJMM2, Proposition 3.3], the comultiplication rule makes $\mathcal{M}_{\mathbf{a}, \mathbf{b}}(u)$ into a $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -module for generic q_1, q_2, u . Moreover, it is an irreducible *lowest weight* module.

- *Representation $\mathcal{M}_{\mathbf{a}, \mathbf{b}}^{p', p}(u)$.*

Assume further that q_1, q_2 are not generic: there exist $p, p' \geq 1$ such that $q_1^a q_2^b = 1$ iff $a = p'c, b = pc$ for some $c \in \mathbb{Z}$. We require that $a_n := p' - 1 - \sum_{i=1}^{n-1} (a_i + 1), b_n := p - 1 - \sum_{i=1}^{n-1} (b_i + 1)$ are non-negative. In this case, the action of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ on $\mathcal{M}_{\mathbf{a}, \mathbf{b}}(u)$ is ill-defined.

Consider a subspace $\mathcal{M}_{\mathbf{a}, \mathbf{b}}^{p', p}(u) \subset F(u_1) \otimes \cdots \otimes F(u_n)$ spanned by $|\lambda^1, \dots, \lambda^n\rangle$ satisfying the same conditions $\lambda_s^i \geq \lambda_{s+b_i}^{i+1} - a_i$, but with $i \leq n$, where $\lambda^{n+1} := \lambda^1$. The comultiplication rule makes it into a $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -module, due to [FFJMM2, Proposition 3.7]. We think of $\mathcal{M}_{\mathbf{a}, \mathbf{b}}^{p', p}(u)$ as “a subquotient of $F(u_1) \otimes \cdots \otimes F(u_n)$ ”. Their characters coincide with the characters of the \mathcal{W}_n -minimal series of \mathfrak{sl}_n -type, according to the main result of [FFJMM2].

4.6. The categories \mathcal{O} . We conclude this section by introducing the appropriate categories \mathcal{O} both for the quantum toroidal and the affine Yangian of \mathfrak{gl}_1 .

- Category \mathcal{O} for $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$.

As we will see in the next sections it is convenient to work with the quotient algebra $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1) := \ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)/(\psi_0^- - (\psi_0^+)^{-1})$, rather than $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ itself. The algebra $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ is graded by $\deg(e_i) = -1, \deg(f_i) = 1, \deg(\psi_j^\pm) = 0$.

Definition 4.2. We say that a \mathbb{Z} -graded $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -module L is in the category \mathcal{O} if

- (i) for any $v \in L$ there exists $N \in \mathbb{Z}$ such that $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)_{\geq N}(v) = 0$,
- (ii) the graded component L_k is finite dimensional for any $k \in \mathbb{Z}$ (module L is of *finite type*).

We say that L is a highest weight module if there exists $v_0 \in L$ generating L and such that $f_i(v_0) = 0, \psi_j^\pm(v_0) = p_j^\pm \cdot v_0, \forall i \in \mathbb{Z}, j \in \mathbb{Z}_+$, for some $p_j^\pm \in \mathbb{C}$ with $p_0^+ \cdot p_0^- = 1$. To such a collection $\{p_j^\pm\}$, we associate two series $p^\pm(z) := \sum_{j \geq 0} p_j^\pm z^{\mp j} \in \mathbb{C}[[z^{\mp 1}]]$. Given any two series $p^+(z), p^-(z)$ satisfying $p_0^+ \cdot p_0^- = 1$, there is a universal highest weight representation M_{p^+, p^-} , which may be defined as the quotient of $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ by the left-ideal generated by $\{f_i\} \cup \{\psi_j^\pm - p_j^\pm\}$. By a standard argument M_{p^+, p^-} has a unique irreducible quotient V_{p^+, p^-} .

The module V_{p^+, p^-} obviously satisfies the condition (i) from the definition of the category \mathcal{O} . Our next result provides a criteria for V_{p^+, p^-} to satisfy (ii) (i.e., to be in the category \mathcal{O}).

Proposition 4.10. *The module V_{p^+, p^-} is of finite type iff there exists a rational function $P(z)$ such that $p^\pm(z) = P(z)^\pm$ and $P(0)P(\infty) = 1$.*

Proof.

The proof is standard and is based on the arguments from [CP]. Define constants $\{p_i\}_{i \in \mathbb{Z}}$ as p_i^+ (for $i > 0$), $-p_i^-$ (for $i < 0$), and $p_0^+ - p_0^-$ (for $i = 0$). To prove the “only if” part we choose indices $k \in \mathbb{Z}, l \in \mathbb{Z}_+$ such that $\{e_k(v_0), \dots, e_{k+l}(v_0)\}$ span the degree -1 component $(V_{p^+, p^-})_{-1}$, while this fails for the collection $\{e_k(v_0), \dots, e_{k+l-1}(v_0)\}$. As a result, there are complex numbers $a_0, \dots, a_l \in \mathbb{C}, a_l \neq 0$, such that $a_0 e_k(v_0) + a_1 e_{k+1}(v_0) + \dots + a_l e_{k+l}(v_0) = 0$. Applying the operator f_{r-k} to this identity and using the equality $f_i e_j(v_0) = -\beta_1^{-1} p_{i+j} \cdot v_0$ (due to (T3)), we get $a_0 p_r + a_1 p_{r+1} + \dots + a_l p_{r+l} = 0$ for all $r \in \mathbb{Z}$. Therefore, the collection $\{p_i\}_{i \in \mathbb{Z}}$ satisfies a simple recurrence relation. Solving this recurrence relation and using the conditions $p_0 = p_0^+ - p_0^-, p_0^- = (p_0^+)^{-1}$, we immediately see that $p^\pm(z)$ are extension in $z^{\mp 1}$ of the same rational function.

To prove the “if” direction, the same arguments show $\dim(V_{p^+, p^-})_{-1} < \infty$. Combining this with the relation (T1) a simple induction argument implies that $\dim(V_{p^+, p^-})_{-l} < \infty \forall l > 0$. \square

- Category \mathcal{O} for $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$.

The algebra $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ is graded by $\deg(e_j) = -1, \deg(f_j) = 1, \deg(\psi_j) = 0$.

Definition 4.3. We say that a \mathbb{Z} -graded $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -module L is in the category \mathcal{O} if

- (i) for any $v \in L$ there exists $N \in \mathbb{Z}$ such that $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)_{\geq N}(v) = 0$,
- (ii) the graded component L_k is finite dimensional for any $k \in \mathbb{Z}$ (module L is of *finite type*).

We say that L is a highest weight module if there exists $v_0 \in L$ generating L and such that $f_j(v_0) = 0, \psi_j(v_0) = p_j \cdot v_0, \forall j \in \mathbb{Z}_+$, for some $p_j \in \mathbb{C}$. Set $p(z) := 1 + \sum_{j \geq 0} p_j z^{-j-1} \in \mathbb{C}[[z^{-1}]]$. For any $\{p_j\}$, there is a universal highest weight $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ -representation M_p , which may be defined as the quotient of $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ by the left-ideal generated by $\{f_j\} \cup \{\psi_j - p_j\}$. It has a unique irreducible quotient V_p . The following is analogous to Proposition 4.10:

Proposition 4.11. *The module V_p is in the category \mathcal{O} iff there exists a rational function $P(z)$ such that $p(z) = P(z)^+$ and $P(\infty) = 1$.*

5. THE LIMIT ALGEBRAS

5.1. The algebras \mathfrak{d}_h and $\bar{\mathfrak{d}}_h$. We recall the algebra of h -difference operators on \mathbb{C} .

For a formal variable h , let \mathfrak{d}_h be an associative algebra over $\mathbb{C}[[h]]$ topologically generated by $Z^{\pm 1}, D^{\pm 1}$ subject to the following relations

$$Z \cdot Z^{-1} = Z^{-1} \cdot Z = 1, \quad D \cdot D^{-1} = D^{-1} \cdot D = 1, \quad D \cdot Z = qZ \cdot D, \quad \text{where } q = \exp(h) \in \mathbb{C}[[h]].$$

We will view \mathfrak{d}_h as a Lie algebra with the natural commutator-Lie bracket $[\cdot, \cdot]$. It is easy to check that the following formula defines a 2-cocycle $\phi_{\mathfrak{d}} \in C^2(\mathfrak{d}_h, \mathbb{C}[[h]])$:

$$\phi_{\mathfrak{d}}(Z^a D^j, Z^b D^{-j'}) = \begin{cases} 0 & j \neq j' \text{ or } j = j' = 0 \\ \sum_{i=-j}^{-1} q^{ai+b(j+i)} & j = j' > 0 \\ -\sum_{i=j}^{-1} q^{bi+a(-j+i)} & j = j' < 0 \end{cases}.$$

This endows $\bar{\mathfrak{d}}_h = \mathfrak{d}_h \oplus \mathbb{C}[[h]] \cdot c_{\mathfrak{d}}$ with a structure of a Lie algebra.

5.2. The algebras \mathfrak{D}_h and $\bar{\mathfrak{D}}_h$. We recall the algebra of q -difference operators on \mathbb{C}^* .

For a formal variable h , let \mathfrak{D}_h be an associative algebra over $\mathbb{C}[[h]]$ topologically generated by $x, \partial^{\pm 1}$ subject to the following defining relations

$$\partial \cdot \partial^{-1} = \partial^{-1} \cdot \partial = 1, \quad \partial \cdot x = (x + h) \cdot \partial.$$

We will view \mathfrak{D}_h as a Lie algebra with the natural commutator-Lie bracket $[\cdot, \cdot]$. It is easy to check that the following formula defines a 2-cocycle $\phi_{\mathfrak{D}} \in C^2(\mathfrak{D}_h, \mathbb{C}[[h]])$:

$$\phi_{\mathfrak{D}}(f(x)\partial^r, g(x)\partial^{-s}) = \begin{cases} 0 & r \neq s \text{ or } r = s = 0 \\ \sum_{l=-r}^{-1} f(lh)g((l+r)h) & r = s > 0 \\ -\sum_{l=r}^{-1} g(lh)f((l-r)h) & r = s < 0 \end{cases}.$$

This endows $\bar{\mathfrak{D}}_h = \mathfrak{D}_h \oplus \mathbb{C}[[h]] \cdot c_{\mathfrak{D}}$ with a structure of a Lie algebra.

5.3. The isomorphism Υ_0 . We construct an isomorphism of the completions of $\bar{\mathfrak{d}}_h$ and $\bar{\mathfrak{D}}_h$.

First we introduce the appropriate completions of the algebras $\bar{\mathfrak{d}}_h, \bar{\mathfrak{D}}_h$:

- $\widehat{\bar{\mathfrak{d}}_h}$ is the completion of $\bar{\mathfrak{d}}_h$ with respect to the powers of the two-sided ideal $J_{\mathfrak{d}} = (Z - 1, q - 1)$;
- $\widehat{\bar{\mathfrak{D}}_h}$ is the completion of $\bar{\mathfrak{D}}_h$ with respect to the powers of the two-sided ideal $J_{\mathfrak{D}} = (x, h)$.

In other words, we have:

$$\widehat{\bar{\mathfrak{d}}_h} := \varprojlim \bar{\mathfrak{d}}_h / \bar{\mathfrak{d}}_h \cdot (Z - 1, q - 1)^j, \quad \widehat{\bar{\mathfrak{D}}_h} := \varprojlim \bar{\mathfrak{D}}_h / \bar{\mathfrak{D}}_h \cdot (x, h)^j.$$

Remark 5.1. Taking completions of \mathfrak{d}_h and \mathfrak{D}_h with respect to the ideals $J_{\mathfrak{d}}$ and $J_{\mathfrak{D}}$ commutes with taking central extensions with respect to the 2-cocycles $\phi_{\mathfrak{d}}$ and $\phi_{\mathfrak{D}}$.

The following result is straightforward:

Proposition 5.1. *There exists an isomorphism $\Upsilon_0 : \widehat{\bar{\mathfrak{d}}_h} \xrightarrow{\sim} \widehat{\bar{\mathfrak{D}}_h}$, defined on the generators by*

$$D^{\pm 1} \mapsto \partial^{\pm 1}, \quad Z^{\pm 1} \mapsto e^{\pm x}, \quad c_{\mathfrak{d}} \mapsto c_{\mathfrak{D}}.$$

Remark 5.2. Specializing h to a complex parameter $h_0 \in \mathbb{C}$, that is taking factor by $(h - h_0)$, we get the classical \mathbb{C} -algebras of difference operators \mathfrak{D}_{h_0} and \mathfrak{d}_{h_0} . However, one can not define their completions as above and, moreover, completions of their central extensions.

5.4. The algebra $\ddot{U}'_h(\mathfrak{gl}_1)$. We introduce the *limit algebra* $\ddot{U}'_h(\mathfrak{gl}_1)$.

Throughout this section, we let h_1, h_2 be formal variables and set $h_3 := -h_1 - h_2$. We define $q_i := \exp(h_i) \in \mathbb{C}[[h_1, h_2]]$ for $i = 1, 2, 3$. First we introduce a *formal* analogue of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$. While the relations (T0, T1, T2, T4t, T5t, T6t) are well-defined over $\mathbb{C}[[h_1, h_2]]$, we need to change (T3) in an appropriate way. This will also lead to *renormalizations* of (T4t, T5t).

Remark 5.3. This is analogous to the classical relation between $U_q(\widehat{\mathfrak{g}})$ and $U_q(L\mathfrak{g})$.

We start by *renormalizing* (T3) to the following form:

$$(T3') \quad [e_i, f_j] = (\psi_{i+j}^+ - \psi_{-i-j}^-)/(1 - q_3).$$

This procedure is called *renormalization*, since for the case of complex parameters $q_i \neq 1$, this algebra is obtained from $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ just by *rescaling* e_i by $1 - q_1$ and f_i by $1 - q_2$.

Next, we write $\psi^\pm(z)$ as $\psi^\pm(z) = \exp(\frac{h_3}{2}\kappa_\pm) \cdot \exp(\pm(1 - q_3) \sum_{\pm m > 0} H_m z^{-m})$. Then κ_\pm are central elements and the relations (T4t, T5t) get modified to:

$$[H_m, e_i] = -\frac{(1 - q_1^m)(1 - q_2^m)(1 - q_3^m)}{m(1 - q_3)} e_{i+m}, \quad [H_m, f_i] = \frac{(1 - q_1^m)(1 - q_2^m)(1 - q_3^m)}{m(1 - q_3)} f_{i+m}.$$

These relations are well-defined in the formal setting since $\frac{(1 - q_1^m)(1 - q_2^m)(1 - q_3^m)}{1 - q_3} \in \mathbb{C}[[h_1, h_2]]$. Note that the right hand side of the relation (T3') also makes sense. The corresponding algebra over $\mathbb{C}[[h_1, h_2]]$ topologically generated by $\{e_i, f_i, \kappa_\pm, H_m\}$ will be denoted by $\ddot{U}_{h_2, h_3}(\mathfrak{gl}_1)$. We also introduce $\ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1) := \ddot{U}_{h_2, h_3}(\mathfrak{gl}_1)/(\kappa_+ + \kappa_-)$. Finally, we define $\ddot{U}'_h(\mathfrak{gl}_1)$ by

$$\ddot{U}'_h(\mathfrak{gl}_1) := \ddot{U}_{h, h_3}(\mathfrak{gl}_1)/(h_3).$$

It is a $\mathbb{C}[[h]]$ -algebra topologically generated by $\{e_j, f_j, \kappa, H_m\}$ subject to (T1, T2, T6) and

$$(T0L) \quad [H_k, H_m] = 0, \quad \kappa \text{ is central,}$$

$$(T3L) \quad [e_i, f_j] = \begin{cases} H_{i+j} & i+j \neq 0 \\ -\kappa & i+j = 0 \end{cases},$$

$$(T4tL) \quad [H_m, e_i] = -(1 - q^m)(1 - q^{-m})e_{i+m},$$

$$(T5tL) \quad [H_m, f_i] = (1 - q^m)(1 - q^{-m})f_{i+m},$$

where $q := \exp(h) \in \mathbb{C}[[h]]$. Now we are ready to relate $\ddot{U}'_h(\mathfrak{gl}_1)$ to $\bar{\mathfrak{d}}_h$.

Proposition 5.2. *There exists a homomorphism $\theta_m : \ddot{U}'_h(\mathfrak{gl}_1) \rightarrow U(\bar{\mathfrak{d}}_h)$ such that*

$$\theta_m(e_i) = Z^i D, \quad \theta_m(f_i) = -D^{-1} Z^i, \quad \theta_m(H_k) = -(1 - q^{-k})Z^k - q^{-k}c_{\mathfrak{d}}, \quad \theta_m(\kappa) = c_{\mathfrak{d}}.$$

Proof.

It suffices to check that $\tilde{e}_i := Z^i D, \tilde{f}_i := -D^{-1} Z^i, \tilde{H}_k := -(1 - q^{-k})Z^k - q^{-k}c_{\mathfrak{d}}, \tilde{\kappa} = c_{\mathfrak{d}}$ satisfy the defining relations of $\ddot{U}'_h(\mathfrak{gl}_1)$. The only nontrivial relations are (T1, T3L, T4tL, T6t).

- For $i, j \in \mathbb{Z}$, we have $[\tilde{e}_i, \tilde{e}_j] = [Z^i D, Z^j D] = (q^j - q^i) \cdot Z^{i+j} D^2$. Therefore, (T1) follows:

$$[\tilde{e}_{n+3}, \tilde{e}_m] - (1 + q + q^{-1})[\tilde{e}_{n+2}, \tilde{e}_{m+1}] + (1 + q + q^{-1})[\tilde{e}_{n+1}, \tilde{e}_{m+2}] - [\tilde{e}_n, \tilde{e}_{m+3}] = 0.$$

- The relation (T3L) follows from the following identity:

$$[\tilde{e}_i, \tilde{f}_j] = -[Z^i D, D^{-1} Z^j] = (-1 + q^{-i-j})Z^{i+j} - q^{-i-j}c_{\mathfrak{d}}.$$

- The relation (T4tL) follows from the following identity:

$$[\tilde{H}_m, \tilde{e}_i] = -(1 - q^{-m})[Z^m, Z^i D] = -(1 - q^{-m})(1 - q^m)Z^{i+m} D = -(1 - q^m)(1 - q^{-m})\tilde{e}_{i+m}.$$

- The relation (T6t) follows from $[D, [ZD, Z^{-1}D]] = [D, (q^{-1} - q)D + q^{-1}c_{\mathfrak{d}}] = 0$. \square

The image of θ_m is easy to describe.

Lemma 5.3. *Let $\bar{\mathfrak{d}}_h^0 \subset \bar{\mathfrak{d}}_h$ be a free $\mathbb{C}[[h]]$ -submodule generated by*

$$\{c_{\mathfrak{d}}, h \cdot Z^k D^0, h^{j-1} Z^i D^j, h^{j-1} Z^i D^{-j} | k \neq 0, j > 0\}.$$

Then $\bar{\mathfrak{d}}_h^0$ is a Lie subalgebra of $\bar{\mathfrak{d}}_h$ and $\text{Im}(\theta_m) \subset U(\bar{\mathfrak{d}}_h^0)$.

Actually, we have the following result:

Theorem 5.4. *The homomorphism θ_m provides an isomorphism $\ddot{U}'_h(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{d}}_h^0)$.*

Note that all the defining relations of $\ddot{U}'_h(\mathfrak{gl}_1)$ are of Lie-type. Hence, $\ddot{U}'_h(\mathfrak{gl}_1)$ is an enveloping algebra of the Lie algebra generated by e_i, f_i, κ, H_m with the aforementioned defining relations. Thus, Theorem 5.4 provides a presentation of the Lie algebra $\bar{\mathfrak{d}}_h^0$ by generators and relations.

Actually, we will prove a more general fact in Appendix C:

Theorem 5.5. *If $h_0 \in \mathbb{C} \setminus \{\mathbb{Q} \cdot \pi i\}$, then θ_m induces an isomorphism of the \mathbb{C} -algebras: $\ddot{U}'_{h_0}(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{d}}'_{h_0})$, where $\bar{\mathfrak{d}}'_{h_0} \subset \bar{\mathfrak{d}}_{h_0}$ is a Lie subalgebra spanned by $c_{\mathfrak{d}}$ and $\{Z^i D^j\}_{(i,j) \neq (0,0)}$.*

5.5. The algebra $\ddot{Y}'_h(\mathfrak{gl}_1)$. We introduce the *limit algebra* $\ddot{Y}'_h(\mathfrak{gl}_1)$

Analogously to the previous section, we let h_1, h_2 be formal variables and set $h_3 := -h_1 - h_2$. We view $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ as a *formal* version of the corresponding algebra introduced in Section 1.3. In other words, $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ is an associative algebra over $\mathbb{C}[[h_1, h_2]]$ topologically generated by $\{e_j, f_j, \psi_j\}_{j \in \mathbb{Z}_+}$ subject to the relations (Y0)-(Y6).

We will actually need a *homogenized version* of this algebra. Let $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$ be an associative algebra over $\mathbb{C}[[h_1, h_2]]$ defined similarly to $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ with the following few changes:

$$[\psi_2, e_i] = -2h_1 h_2 e_i, \quad [\psi_2, f_i] = 2h_1 h_2 f_i.$$

The specializations of algebras $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ at $h_k \in \mathbb{C}^*$ are isomorphic. However, $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$ is a \mathbb{Z}_+ -graded algebra with $\deg(e_i) = i, \deg(f_i) = i, \deg(\psi_i) = i, \deg(h_k) = 1$.

We define $\ddot{Y}'_h(\mathfrak{gl}_1)$ by

$$\ddot{Y}'_h(\mathfrak{gl}_1) := \ddot{Y}'_{h, h_3}(\mathfrak{gl}_1) / (h_3).$$

It is an algebra over $\mathbb{C}[[h]]$. The following result is straightforward:

Proposition 5.6. *There exists a homomorphism $\theta_a : \ddot{Y}'_h(\mathfrak{gl}_1) \rightarrow U(\bar{\mathfrak{D}}_h)$ such that*

$$\theta_a(e_j) = x^j \partial, \quad \theta_a(f_j) = -\partial^{-1} x^j, \quad \theta_a(\psi_j) = (x - h)^j - x^j - (-h)^j c_{\mathfrak{D}}.$$

The image of θ_a is easy to describe.

Lemma 5.7. *Let $\bar{\mathfrak{D}}_h^0 \subset \bar{\mathfrak{D}}_h$ be a free $\mathbb{C}[[h]]$ -submodule generated by*

$$\{c_{\mathfrak{D}}, h \cdot x^i \partial^0, h^{j-1} x^i \partial^j, h^{j-1} x^i \partial^{-j} | i \geq 0, j > 0\}.$$

Then $\bar{\mathfrak{D}}_h^0$ is a Lie subalgebra of $\bar{\mathfrak{D}}_h$ and $\text{Im}(\theta_a) \subset U(\bar{\mathfrak{D}}_h^0)$.

Actually, we have the following result:

Theorem 5.8. *The homomorphism θ_a provides an isomorphism $\theta_a : \ddot{Y}'_h(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{D}}_h^0)$.*

Note that all the defining relations of $\ddot{Y}'_h(\mathfrak{gl}_1)$ are of Lie-type. Hence, $\ddot{Y}'_h(\mathfrak{gl}_1)$ is an enveloping algebra of the Lie algebra generated by e_j, f_j, ψ_j with the aforementioned defining relations. Thus, Theorem 5.8 provides a presentation of the Lie algebra $\bar{\mathfrak{D}}_h^0$ by generators and relations.

Actually, we will prove a more general fact in Appendix C:

Theorem 5.9. *For $h_i \in \mathbb{C}^*$, θ_a induces an isomorphism of \mathbb{C} -algebras $\theta_a : \ddot{Y}'_{h_0}(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{D}}_{h_0}^0)$.*

6. THE HOMOMORPHISM Υ

We construct a homomorphism $\Upsilon : \ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1) \rightarrow \hat{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$, which induces an inclusion (but not an isomorphism as it was in [GTL]) of appropriate completions. We also construct compatible homomorphisms $\text{ch}_r : M^r \rightarrow \widehat{V}^r$.

6.1. Construction of Υ . We follow notation of [GTL].

Let $\hat{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$ denote the completion of $\ddot{Y}_{h_2, h_3}(\mathfrak{gl}_1)$ with respect to the \mathbb{Z}_+ -grading on it. To state the main result, we introduce the following notation:

- Define $\psi(z) := 1 - h_3 \sum_{i \geq 0} \psi_i z^{-i-1} \in \ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)[[z^{-1}]]$ (agrees with that from Section 1.4).
- Define $k_i \in \mathbb{C}[\psi_0, \psi_1, \psi_2, \dots]$ by $\sum_{i \geq 0} k_i z^{-i-1} =: k(z) = \ln(\psi(z))$.
- Define the *inverse Borel transform* $B : z^{-1}\mathbb{C}[[z^{-1}]] \rightarrow \mathbb{C}[[w]]$ by $\sum_{i=0}^{\infty} \frac{a_i}{z^{i+1}} \mapsto \sum_{i=0}^{\infty} \frac{a_i}{i!} w^i$.
- Define $B(w) \in h_3 \ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)[[z^{-1}]]$ to be the inverse Borel transform of $k(z)$.
- Define a function $G(v) := \log\left(\frac{v}{e^{v/2} - e^{-v/2}}\right) \in v\mathbb{Q}[[v]]$.
- Define $\gamma(v) := -B(-\partial_v)G'(v) \in \hat{Y}'_{h_2, h_3}(\mathfrak{gl}_1)[[v]]$.
- Define $g(v) := \sum_{i \geq 0} g_i v^i \in \hat{Y}'_{h_2, h_3}(\mathfrak{gl}_1)[[v]]$ by $g(v) := \left(\frac{h_3}{q_3 - 1}\right)^{1/2} \exp\left(\frac{\gamma(v)}{2}\right)$.

The identity $B(\log(1 - \gamma/z)) = (1 - e^{\gamma w})/w$ immediately implies the following result:

Corollary 6.1. *The conditions of Proposition 1.3(e,f) are equivalent to*

$$[B(w), e_j] = \frac{\sum_{i=1}^3 (e^{h_i w} - e^{-h_i w})}{w} e^{w\sigma^+} e_j, \quad [B(w), f_j] = \frac{\sum_{i=1}^3 (e^{-h_i w} - e^{h_i w})}{w} e^{w\sigma^-} f_j.$$

Now we are ready to state the main result of this section:

Theorem 6.2. *There exists an algebra homomorphism*

$$\Upsilon : \ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1) \rightarrow \hat{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$$

defined on the generators by

$$H_m \mapsto \frac{B(m)}{1 - q_3}, \quad e_k \mapsto e^{k\sigma^+} g(\sigma^+) e_0, \quad f_k \mapsto e^{k\sigma^-} g(\sigma^-) f_0, \quad \kappa \mapsto -\psi_0.$$

Proof.

We need to verify that Υ is compatible with the defining relations of $\ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1)$.

- The relation (T0) is obviously preserved by Υ .
- According to Corollary 6.1, we have

$$[B(m), e_j] = \frac{q_1^m + q_2^m + q_3^m - q_1^{-m} - q_2^{-m} - q_3^{-m}}{m} e^{m\sigma^+} e_j = -\frac{(1 - q_1^m)(1 - q_2^m)(1 - q_3^m)}{m} e^{m\sigma^+} e_j,$$

$$[B(m), f_j] = -\frac{q_1^m + q_2^m + q_3^m - q_1^{-m} - q_2^{-m} - q_3^{-m}}{m} e^{m\sigma^-} f_j = \frac{(1 - q_1^m)(1 - q_2^m)(1 - q_3^m)}{m} e^{m\sigma^-} f_j.$$

This implies the compatibility of (T4t, T5t) with Υ .

- The verification of (T1, T2, T3) is completely analogous to the corresponding computations from [GTL, Ch. 3.4].
- The verification of the cubic relation (T6t) is implicit. Define $E := [\Upsilon(e_0), [\Upsilon(e_1), \Upsilon(e_{-1})]]$. We will see (Proposition 6.8 below) that E acts trivially on V^r for all r . Note that V^r are \mathbb{Z}_+ -graded modules of $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$. In particular, the degree k component E_k of E acts trivially on V^r . But we will see (Section 6.4 below) that the action of $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$ on $\bigoplus_r V^r$ is faithful. This implies $E_k = 0$ for all k , and $E = 0$. The proof of $[\Upsilon(f_0), [\Upsilon(f_1), \Upsilon(f_{-1})]] = 0$ is analogous. \square

6.2. The limit $h_3 = 0$. We verify that the specialization of Υ at $h_3 = 0$ is induced by Υ_0 .

Recall that we have isomorphisms

$$\ddot{U}'_{h,h_3}(\mathfrak{gl}_1)/(h_3) \xrightarrow{\sim} U(\bar{\mathfrak{d}}_h^0) \quad \text{and} \quad \ddot{Y}'_{h,h_3}(\mathfrak{gl}_1)/(h_3) \xrightarrow{\sim} U(\bar{\mathfrak{d}}_h^0).$$

Our next result evaluates the specialization of Υ at $h_3 = 0$.

Proposition 6.3. *The limit homomorphism $\Upsilon|_{h_3=0} : U(\bar{\mathfrak{d}}_h^0) \rightarrow \widehat{U(\bar{\mathfrak{d}}_h^0)}$ is induced by Υ_0 .*

Proof.

We verify the statement by computing the images of the generators under $\Upsilon|_{h_3=0}$. We have:

- $\Upsilon|_{h_3=0}(c_{\mathfrak{d}}) = c_{\mathfrak{D}}$.
- $\Upsilon|_{h_3=0}((q^{-k} - 1)Z^k - q^{-k}c_{\mathfrak{d}}) = \sum_{i \geq 0} ((x - h)^i - x^i - (-h)^i c_{\mathfrak{D}}) \frac{k^i}{i!} = (q^{-k} - 1)e^{kx} - q^{-k}c_{\mathfrak{D}}$.
- $\Upsilon|_{h_3=0}(Z^k D) = \sum_{i \geq 0} \frac{k^i}{i!} \cdot x^i \partial = e^{kx} \partial$.
- $\Upsilon|_{h_3=0}(-D^{-1}Z^k) = -\sum_{i \geq 0} \frac{k^i}{i!} \partial^{-1} \cdot x^i = -\partial^{-1}e^{kx}$.

The result follows. \square

6.3. The elliptic Hall algebra. We recall a notion of the elliptic Hall algebra studied in [BS].

We will need the following notation:

- We set $(\mathbb{Z}^2)^* := \mathbb{Z}^2 \setminus \{(0, 0)\}$, $(\mathbb{Z}^2)^+ := \{(a, b) | a > 0 \text{ or } a = 0, b > 0\}$, $(\mathbb{Z}^2)^- := -(\mathbb{Z}^2)^+$.
- For any $\mathbf{x} = (a, b) \in (\mathbb{Z}^2)^*$, we define $\deg(\mathbf{x}) := \gcd(a, b)$.
- For any $\mathbf{x} \in (\mathbb{Z}^2)^*$, we define $\epsilon_{\mathbf{x}} := 1$ if $\mathbf{x} \in (\mathbb{Z}^2)^+$ and $\epsilon_{\mathbf{x}} := -1$ if $\mathbf{x} \in (\mathbb{Z}^2)^-$.
- For a pair of non-collinear $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}^2)^*$, we set $\epsilon_{\mathbf{x}, \mathbf{y}} := \text{sign}(\det(\mathbf{x}, \mathbf{y})) \in \{\pm 1\}$.
- For non-collinear $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}^2)^*$, we denote the triangle with vertices $\{(0, 0), \mathbf{x}, \mathbf{x} + \mathbf{y}\}$ by $\Delta_{\mathbf{x}, \mathbf{y}}$.
- We define $\alpha_n := -\frac{\beta_n}{n} = \frac{(1-q_1^{-n})(1-q_2^{-n})(1-q_3^{-n})}{n}$.
- We say that $\Delta_{\mathbf{x}, \mathbf{y}}$ is *empty* if there are no lattice points inside this triangle.

Following [BS], we define (central extension of) the elliptic Hall algebra $\tilde{\mathcal{E}}$ to be the associative algebra generated by $\{u_{\mathbf{x}}, \kappa_{\mathbf{y}} | \mathbf{x} \in (\mathbb{Z}^2)^*, \mathbf{y} \in \mathbb{Z}^2\}$ with the following defining relations:

$$(E0) \quad \kappa_{\mathbf{x}} \kappa_{\mathbf{y}} = \kappa_{\mathbf{x} + \mathbf{y}}, \quad \kappa_{0,0} = 1,$$

$$(E1) \quad [u_{\mathbf{y}}, u_{\mathbf{x}}] = \delta_{\mathbf{x}, -\mathbf{y}} \cdot \frac{\kappa_{\mathbf{x}} - \kappa_{\mathbf{x}}^{-1}}{\alpha_{\deg(\mathbf{x})}} \text{ if } \mathbf{x}, \mathbf{y} \text{ are collinear},$$

$$(E2) \quad [u_{\mathbf{y}}, u_{\mathbf{x}}] = \epsilon_{\mathbf{x}, \mathbf{y}} \kappa_{\alpha(\mathbf{x}, \mathbf{y})} \frac{\theta_{\mathbf{x} + \mathbf{y}}}{\alpha_1} \text{ if } \Delta_{\mathbf{x}, \mathbf{y}} \text{ is empty and } \deg(\mathbf{x}) = 1,$$

where the elements $\theta_{\mathbf{x}}$ are defined via

$$(E3) \quad \sum_{n \geq 0} \theta_{n\mathbf{x}_0} x^n = \exp \left(\sum_{r > 0} \alpha_r u_{r\mathbf{x}_0} x^r \right) \text{ if } \deg(\mathbf{x}_0) = 1,$$

while $\alpha(\mathbf{x}, \mathbf{y})$ is defined by

$$(E4) \quad \alpha(\mathbf{x}, \mathbf{y}) = \begin{cases} \epsilon_{\mathbf{x}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x} + \mathbf{y}}(\mathbf{x} + \mathbf{y}))/2, & \epsilon_{\mathbf{x}, \mathbf{y}} = 1 \\ \epsilon_{\mathbf{y}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x} + \mathbf{y}}(\mathbf{x} + \mathbf{y}))/2, & \epsilon_{\mathbf{x}, \mathbf{y}} = -1 \end{cases}.$$

This algebra is closely related to the toroidal algebras of \mathfrak{gl}_1 :

Theorem 6.4. [S] *There is an isomorphism $\Xi : \tilde{\mathcal{E}}/(\kappa_{0,1} - 1) \xrightarrow{\sim} \ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ defined on the generators by*

$$u_{1,i} \mapsto e_i, \quad u_{-1,i} \mapsto f_i, \quad \theta_{0,j} \mapsto \psi_j^+ / \psi_0^+, \quad \theta_{0,-j} \mapsto \psi_j^- / \psi_0^-, \quad \kappa_{a,b} \mapsto (\psi_0^+)^a, \quad j > 0.$$

Remark 6.1. This theorem has been proved in [S] only for $\mathcal{E} := \tilde{\mathcal{E}}/(\kappa_{\mathbf{y}} - 1)_{\mathbf{y} \in \mathbb{Z}^2}$, but the above generalization is straightforward. The quotient algebra \mathcal{E} is the spherical Hall algebra of an elliptic curve over \mathbb{F}_q .

This result provides distinguished elements $\{u_{\mathbf{x}} | \mathbf{x} \in (\mathbb{Z}^2)^*\}$ of $\ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1)$. As $h_3 \rightarrow 0$, their images $\bar{u}_{\mathbf{x}}$ coincide with the natural generators of $U(\bar{\mathfrak{d}}_{h_2}^0)$.

Lemma 6.5. *The θ_m -images of $\bar{u}_{k,l}$ are given by the following formulas:*

$$(8) \quad \bar{u}_{0,r} \mapsto \text{sign}(r) \frac{1 - q_2^{-1}}{1 - q_2^r} (1 - q_2) \left(Z^r - \frac{1 - q_2}{1 - q_2^r} c_{\mathfrak{d}} \right),$$

$$(9) \quad \bar{u}_{k,l} \mapsto q_2^{f(k,l)} \frac{1 - q_2^{-1}}{1 - q_2^d} (1 - q_2)^{k-1} Z^l D^k, \quad \bar{u}_{-k,-l} \mapsto -q_2^{-f(k,l)} \frac{1 - q_2}{1 - q_2^d} (1 - q_2^{-1})^{k-1} D^{-k} Z^{-l},$$

where $k > 0$, $r \neq 0$, $d := \gcd(k, l)$ and $f(k, l) := \frac{kl - k - l - d + 2}{2}$ is the (signed) number of lattice points inside the triangle with vertices $\{(0, 0), (0, l), (k, l)\}$.

Proof.

Let us first observe that in the limit $h_3 \rightarrow 0$, the relation (E2) becomes

$$(E2') \quad [\bar{u}_{\mathbf{y}}, \bar{u}_{\mathbf{x}}] = \epsilon_{\mathbf{x}, \mathbf{y}} \kappa_{\alpha(\mathbf{x}, \mathbf{y})} \frac{\alpha_{\deg(\mathbf{x} + \mathbf{y})}}{\alpha_1} \bar{u}_{\mathbf{x} + \mathbf{y}} \text{ if } \Delta_{\mathbf{x}, \mathbf{y}} \text{ is empty and } \deg(\mathbf{x}) = 1.$$

This formula immediately implies (8), since we have the equality $\bar{u}_{0,r} = \text{sign}(r) \frac{\alpha_1}{\alpha_r} [\bar{u}_{-1,0}, \bar{u}_{1,r}]$.

Formula (9) will be proved by an induction on k ; we will consider only the case $k > 0$. Case $k = 1$ is trivial. Given $(k, l) \in \mathbb{Z}_{>1} \times \mathbb{Z}$, choose unique $\mathbf{x} = (k_1, l_1)$, $\mathbf{y} = (k_2, l_2)$, $0 < k_1, k_2 < k$, such that $\mathbf{x} + \mathbf{y} = (k, l)$, $\epsilon_{\mathbf{x}, \mathbf{y}} = 1$, $\deg(\mathbf{x}) = \deg(\mathbf{y}) = 1$ and $\Delta_{\mathbf{x}, \mathbf{y}}$ is empty. The formula (E2') together with an induction assumption yield:

$$\theta_m(\bar{u}_{k,l}) = \frac{(1 - q_2)(1 - q_2^{-1})}{(1 - q_2^d)(1 - q_2^{-d})} q_2^{f(k_1, l_1) + f(k_2, l_2)} (q_2^{k_2 l_1} - q_2^{k_1 l_2}) (1 - q_2)^{k_1 + k_2 - 2} Z^{l_1 + l_2} D^{k_1 + k_2}.$$

By our assumptions on \mathbf{x}, \mathbf{y} and the Pick's formula, we get $q_2^{k_2 l_1} - q_2^{k_1 l_2} = q_2^{k_2 l_1} (1 - q_2^d)$. It remains to use the equality $f(k_1, l_1) + f(k_2, l_2) + k_2 l_1 = f(k_1 + k_2, l_1 + l_2) = f(k, l)$, which obviously follows from the combinatorial meaning of f . \square

6.4. Flatness of the deformations.

We prove the following result:

Theorem 6.6. (a) *The algebra $\ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1)$ is a flat deformation of $\ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1)/(h_3) \simeq U(\bar{\mathfrak{d}}_{h_2}^0)$.*
(b) *The algebra $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$ is a flat deformation of $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)/(h_3) \simeq U(\bar{\mathfrak{d}}_{h_2}^0)$.*

As an immediate consequence of this theorem and Proposition 6.3, we get:

Corollary 6.7. *The homomorphism Υ is injective.*

Remark 6.2. We do not get an isomorphism of the appropriate completions (as it was in [GTL]), since the limit homomorphism $\Upsilon|_{h_3=0}$ does not extend to an isomorphism of completions.

To prove Theorem 6.6 it suffices to provide a faithful $U(\bar{\mathfrak{d}}_{h_2}^0)$ -representation (respectively $U(\bar{\mathfrak{d}}_{h_2}^0)$ -representation) which admits a flat deformation to a representation of $\ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1)$ (respectively $\ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1)$). To make use of the representations constructed in Sections 3 and 4, we should work not over $\mathbb{C}[[h_2, h_3]]$, but rather over the localization ring R of $\mathbb{C}[[h_2, h_3]]$ by the homogeneous polynomials in h_2, h_3 . Therefore, we will switch to the extension algebras

$$\ddot{U}'_R(\mathfrak{gl}_1) := \ddot{U}'_{h_2, h_3}(\mathfrak{gl}_1) \otimes_{\mathbb{C}[[h_2, h_3]]} R, \quad \ddot{Y}'_R(\mathfrak{gl}_1) := \ddot{Y}'_{h_2, h_3}(\mathfrak{gl}_1) \otimes_{\mathbb{C}[[h_2, h_3]]} R.$$

Let \mathfrak{gl}_∞ be a Lie algebra of matrices $A = \sum_{i,j \in \mathbb{Z}} a_{i,j} E_{i,j}$ such that $a_{i,j} = 0$ for $|i - j| \gg 0$. Let $\mathfrak{gl}_{\infty, \kappa} = \mathfrak{gl}_\infty \oplus \mathbb{C} \cdot \kappa$ be the central extension of this Lie algebra by the 2-cocycle

$$\phi_{\mathfrak{gl}} \left(\sum a_{i,j} E_{i,j}, \sum b_{i,j} E_{i,j} \right) = \sum_{i \leq 0 < j} a_{i,j} b_{j,i} - \sum_{j \leq 0 < i} a_{i,j} b_{j,i}.$$

For any $u \in \mathbb{C}^*$, consider the homomorphism $\tau_u : U(\bar{\mathfrak{d}}_{h_2}^0)_R \rightarrow U(\mathfrak{gl}_{\infty, \kappa})_R$ such that

$$Z^m D \mapsto - \sum_i u^m q_2^{im} E_{i+1,i}, \quad D^{-1} Z^m \mapsto - \sum_i u^m q_2^{im} E_{i,i+1},$$

$$Z^m \mapsto - \sum_i u^m q_2^{im} E_{i,i} - \frac{u^m - q_2^{-m}}{1 - q_2^{-m}} \kappa, \quad c_{\mathfrak{d}} \mapsto -\kappa.$$

Let $\varpi_u : \check{U}'_R(\mathfrak{gl}_1) \rightarrow U(\mathfrak{gl}_{\infty, \kappa})_R$ be the composition of $\check{U}'_R(\mathfrak{gl}_1) \rightarrow U(\bar{\mathfrak{d}}_{h_2}^0)_R$ and τ_u . Then

$$\varpi_u(e(z)) = - \sum_i E_{i+1,i} \delta(q_2^i u/z), \quad \varpi_u(f(z)) = \sum_i E_{i,i+1} \delta(q_2^i u/z).$$

Let V_∞ be the basic representation of $\mathfrak{gl}_{\infty, \kappa}$. It is realized on $\wedge^{\infty/2} \mathbb{C}^\infty$ with the highest weight vector $v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$ (here \mathbb{C}^∞ is a \mathbb{C} -vector spaces with the basis $\{v_i\}_{i \in \mathbb{Z}}$). Comparing the formulas for the Fock module $F(u)$ with those for the $\mathfrak{gl}_{\infty, \kappa}$ -action on V_∞ , we see that $F(u)$ degenerates to the module $\tau_u^*(V_\infty)$.

It remains to prove that the module $\bigoplus_n \bigoplus_{u_1, \dots, u_n} \tau_{u_1}^*(V_\infty) \otimes \dots \otimes \tau_{u_n}^*(V_\infty)$ is a faithful representation of $U(\mathfrak{gl}_{\infty, \kappa})$. To prove this, we consider a further degeneration as $h_2 \rightarrow 0$. The algebra $\bar{\mathfrak{d}}_0$ is a central extension of the commutative Lie algebra with the basis $\{Z^k D^l\}$. Note that τ_u degenerates as well to provide a homomorphism $\tau_{u,0} : U(\bar{\mathfrak{d}}_0^0)_R \rightarrow U(\mathfrak{gl}_{\infty, \kappa})_R$ defined by $Z^k D^l \mapsto -u^k \sum_i E_{i+1,i}$, $c_{\mathfrak{d}} \mapsto -\kappa$. The image of this homomorphism is just $U(\mathfrak{h})_R$, where \mathfrak{h} is the Heisenberg algebra. It is easy to see that

$$\bigoplus_n \bigoplus_{u_1, \dots, u_n} \tau_{u_1,0}^*(V_\infty) \otimes \dots \otimes \tau_{u_n,0}^*(V_\infty)$$

is a faithful representation of $U(\mathfrak{h})_R$. This completes the proof.

For the $\check{Y}'_R(\mathfrak{gl}_1)$ case, we use the homomorphism $\varsigma_u : U(\bar{\mathfrak{d}}_{h_2}^0)_R \rightarrow U(\mathfrak{gl}_{\infty, \kappa})_R$ defined by

$$x^n \partial \mapsto - \sum_i (u + i h_2)^n E_{i+1,i}, \quad \partial^{-1} x^n \mapsto \sum_i (u + i h_2)^n E_{i,i+1}, \quad x^n \mapsto \sum_i (u + i h_2)^n E_{i,i} + c_n \kappa, \quad c_{\mathfrak{d}} \mapsto \kappa,$$

where $c_n \in R$ are determined recursively by $\binom{n}{1} h_2 c_{n-1} - \binom{n}{2} h_2^2 c_{n-2} + \dots + (-1)^{n+1} h_2^n c_0 + (-h_2)^n - u^n = 0$. The rest of the arguments are the same.

6.5. The homomorphism ch_r . We construct a map $M^r \rightarrow \widehat{V}^r$ compatible with Υ .

Recall the representations M^r and V^r of $\check{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ and $\check{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ from Section 3, which are defined over the fields $\mathbb{C}(\chi_1, \dots, \chi_r, q_1, q_2, q_3)$ and $\mathbb{C}(x_1, \dots, x_r, h_1, h_2, h_3)$ respectively. Let us denote the corresponding representations of $\check{U}'_R(\mathfrak{gl}_1)$ and $\check{Y}'_R(\mathfrak{gl}_1)$ by M'_R and V'_R (here we set $\chi_i = \exp(x_i)$ and also renormalize $e(z), f(z)$ actions appropriately as discussed in Section 5). Both representations have a basis parametrized by r -partitions $\{\bar{\lambda}\}$. The following result is straightforward:

Proposition 6.8. *There exists a unique collection of constants $\{b_{\bar{\lambda}}\}$ from R such that $b_{\bar{0}} = 1$ and the linear map $\text{ch}_r : M'_R \rightarrow \widehat{V}_R^r$ defined by $[\bar{\lambda}] \mapsto b_{\bar{\lambda}} \cdot [\bar{\lambda}]$ satisfies the property*

$$\text{ch}_r(Xv) = \Upsilon(X) \text{ch}_r(v), \quad \forall X \in \check{U}'_R(\mathfrak{gl}_1), v \in M'_R.$$

7. THE SMALL SHUFFLE ALGEBRAS S^m AND S^a

We introduce the *small multiplicative* and *additive shuffle algebras*. We explain their relation to $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$. We also discuss their interesting commutative subalgebras.

7.1. The shuffle algebra S^m . We introduce the small multiplicative shuffle algebra S^m .

Let us consider a \mathbb{Z}_+ -graded \mathbb{C} -vector space $\mathbb{S}^m = \bigoplus_{n \geq 0} \mathbb{S}_n^m$, where \mathbb{S}_n^m consists of rational functions $\frac{f(x_1, \dots, x_n)}{\Delta(x_1, \dots, x_n)^2}$ with $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathfrak{S}_n}$ and $\Delta(x_1, \dots, x_n) := \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

Define the star-product $\star^m : \mathbb{S}_i^m \times \mathbb{S}_j^m \rightarrow \mathbb{S}_{i+j}^m$ by

$$(F \star^m G)(x_1, \dots, x_{i+j}) := \text{Sym}_{\mathfrak{S}_{n+m}} \left(F(x_1, \dots, x_i) G(x_{i+1}, \dots, x_{i+j}) \prod_{\substack{l > i \\ k \leq i}} \omega^m(x_l, x_k) \right)$$

with

$$\omega^m(x, y) := \frac{(x - q_1 y)(x - q_2 y)(x - q_3 y)}{(x - y)^3}.$$

This endows \mathbb{S}^m with a structure of an associative unital \mathbb{C} -algebra.

We say that an element $\frac{f(x_1, \dots, x_n)}{\Delta(x_1, \dots, x_n)^2} \in \mathbb{S}^m$ satisfies the *wheel condition* if $f(x_1, \dots, x_n) = 0$ for any $\{x_1, \dots, x_n\} \subset \mathbb{C}$ such that $x_1/x_2 = q_i, x_2/x_3 = q_j, i \neq j$. Let $S^m \subset \mathbb{S}^m$ be a \mathbb{Z}_+ -graded subspace, consisting of all such elements. The subspace S^m is closed with respect to \star^m .

Definition 7.1. The algebra (S^m, \star^m) is called the *small multiplicative shuffle algebra*.

Recall that q_1, q_2, q_3 are *generic* if $q_1^a q_2^b q_3^c = 1 \iff a = b = c$. We have the following result:

Theorem 7.1. [N, Proposition 3.5] *The algebra S^m is generated by S_1^m for generic q_1, q_2, q_3 .*

The connection of the shuffle algebra S^m to the Hall algebra $\tilde{\mathcal{E}}$ was established in [SV]:

Proposition 7.2. [SV] *The map $u_{1,i} \mapsto x_1^i$ extends to an injective homomorphism $\tilde{\mathcal{E}}^+ \rightarrow S^m$, where $\tilde{\mathcal{E}}^+$ is the subalgebra of $\tilde{\mathcal{E}}$ generated by $\{u_{i,j}\}_{i>0}$.*

Combining this result with Theorems 7.1 and 6.4, we get:

Theorem 7.3. *The algebras $\tilde{\mathcal{E}}^+, \ddot{U}_{q_1, q_2, q_3}^+(\mathfrak{gl}_1), S^m$ are isomorphic.*

7.2. The commutative subalgebra $\mathcal{A}^m \subset S^m$. We recall an interesting subalgebra \mathcal{A}^m .

Following [FHHSY], we introduce an important \mathbb{Z}_+ -graded subspace $\mathcal{A}^m = \bigoplus_{n \geq 0} \mathcal{A}_n^m$ of S^m . Its degree n component is defined by $\mathcal{A}_n^m = \{F \in S_n^m \mid \partial^{(0,k)} F = \partial^{(\infty,k)} F \ \forall \ 0 \leq k \leq n\}$, where $\partial^{(0,k)} F := \lim_{\xi \rightarrow 0} F(x_1, \dots, \xi \cdot x_{n-k+1}, \dots, \xi \cdot x_n)$, $\partial^{(\infty,k)} F := \lim_{\xi \rightarrow \infty} F(x_1, \dots, \xi \cdot x_{n-k+1}, \dots, \xi \cdot x_n)$.

This subspace satisfies the following properties:

Theorem 7.4. [FHHSY, Section 2] *We have:*

- (a) *Suppose $F \in S_n^m$ and $\partial^{(\infty,k)} F$ exist for all $1 \leq k \leq n$, then $F \in \mathcal{A}_n^m$.*
- (b) *The subspace $\mathcal{A}^m \subset S^m$ is \star^m -commutative.*
- (c) *\mathcal{A}^m is \star^m -closed and it is a polynomial algebra in $\{K_j^m\}_{j \geq 1}$ with $K_j^m \in S_j^m$ defined by:*

$$K_1^m(x_1) = x_1^0, \quad K_2^m(x_1, x_2) = \frac{(x_1 - q_1 x_2)(x_2 - q_1 x_1)}{(x_1 - x_2)^2}, \quad K_n^m(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} K_2^m(x_i, x_j).$$

Remark 7.1. The aforementioned elements K_j^m played a crucial role in [FT1]. They were used to construct an action of the Heisenberg algebra on the vector space M from Section 2.2.

Our next result provides an alternative choice for generators of the algebra \mathcal{A}^m , expressed explicitly via S_1^m . We use the following notation: $[P, Q]_m = P \overset{m}{\star} Q - Q \overset{m}{\star} P$ for $P, Q \in S^m$.

Proposition 7.5. *The algebra \mathcal{A}^m is a polynomial algebra in the generators $\{L_j^m\}_{j \geq 1}$ defined by*

$$L_1^m(x_1) = x_1^0 \quad \text{and} \quad L_j^m = \underbrace{[x^1, [x^0, [x^0, \dots, [x^0, x^{-1}]_m \dots]_m]_m]_m}_{j \text{ factors}} \in S_j^m \quad \text{for } j \geq 2.$$

We refer the reader to Appendix D for the proof of this result.

7.3. The shuffle algebra S^a . We introduce an analogous additive shuffle algebra S^a .

Let us consider a \mathbb{Z}_+ -graded \mathbb{C} -vector space $\mathbb{S}^a = \bigoplus_{n \geq 0} \mathbb{S}_n^a$, where \mathbb{S}_n^a consists of rational functions $\frac{f(x_1, \dots, x_n)}{\Delta(x_1, \dots, x_n)^2}$ with $f \in \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$. Define the star-product $\overset{a}{\star} : \mathbb{S}_i^a \times \mathbb{S}_j^a \rightarrow \mathbb{S}_{i+j}^a$ by

$$(F \overset{a}{\star} G)(x_1, \dots, x_{i+j}) := \text{Sym}_{\mathfrak{S}_{n+m}} \left(F(x_1, \dots, x_i) G(x_{i+1}, \dots, x_{i+j}) \prod_{\substack{l > i \\ k \leq i}} \omega^a(x_l, x_k) \right)$$

with

$$\omega^a(x, y) := \frac{(x - y - h_1)(x - y - h_2)(x - y - h_3)}{(x - y)^3}.$$

This endows \mathbb{S}^a with a structure of an associative unital \mathbb{C} -algebra.

We say that an element $\frac{f(x_1, \dots, x_n)}{\Delta(x_1, \dots, x_n)^2} \in \mathbb{S}^a$ satisfies the *wheel condition* if $f(x_1, \dots, x_n) = 0$ for any $\{x_1, \dots, x_n\} \subset \mathbb{C}$ such that $x_1 - x_2 = h_i, x_2 - x_3 = h_j, i \neq j$. Let $S^a \subset \mathbb{S}^a$ be a \mathbb{Z}_+ -graded subspace, consisting of all such elements. The subspace S^a is closed with respect to $\overset{a}{\star}$.

Definition 7.2. The algebra $(S^a, \overset{a}{\star})$ is called the *small additive shuffle algebra*.

The following result is proved analogously to Theorem 7.1.

Theorem 7.6. *For generic h_1, h_2, h_3 ($ah_1 + bh_2 + ch_3 = 0 \iff a = b = c$), the map $e_i \mapsto x_1^i$ extends to an isomorphism $\check{Y}_{h_1, h_2, h_3}^+(\mathfrak{gl}_1) \xrightarrow{\sim} S^a$. In particular, S^a is generated by S_1^a .*

7.4. The commutative subalgebra $\mathcal{A}^a \subset S^a$. We construct an *additive version* of \mathcal{A}^m .

Let us introduce a \mathbb{Z}_+ -graded subspace $\mathcal{A}^a = \bigoplus_{n \geq 0} \mathcal{A}_n^a$ of S^a . Its degree n component \mathcal{A}_n^a consists of those $F \in S_n^a$ such that the limit

$$\partial^{(\infty, k)} F := \lim_{\xi \rightarrow \infty} F(x_1, \dots, x_{n-k}, x_{n-k+1} + \xi, \dots, x_n + \xi)$$

exists for every $1 \leq k \leq n$. The following is an *additive counterpart* of Theorem 7.4:

Theorem 7.7. *We have:*

- (a) *The subspace $\mathcal{A}^a \subset S^a$ is $\overset{a}{\star}$ -commutative.*
- (b) *\mathcal{A}^a is $\overset{a}{\star}$ -closed and it is a polynomial algebra in $\{K_j^a\}_{j \geq 1}$ with $K_j^a \in S_j^a$ defined by:*

$$K_1^a(x_1) = x_1^0, \quad K_2^a(x_1, x_2) = \frac{(x_1 - x_2 - h_1)(x_2 - x_1 - h_1)}{(x_1 - x_2)^2}, \quad K_n^a(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} K_2^a(x_i, x_j).$$

Analogously to Proposition 7.5, the commutative subalgebra \mathcal{A}^a admits an alternative set of generators expressed via S_1^a . Define $[P, Q]_a := P \overset{a}{\star} Q - Q \overset{a}{\star} P$ for $P, Q \in S^a$.

Proposition 7.8. *The algebra \mathcal{A}^a is a polynomial algebra in the generators $\{L_j^a\}_{j \geq 1}$ defined by*

$$L_1^a(x_1) = x_1^0 \quad \text{and} \quad L_j^a = \underbrace{[x^0, [x^0, \dots, [x^0, x^{j-1}]_a \dots]_a]_a}_{j \text{ factors}} \in S_j^a \quad \text{for } j \geq 2.$$

8. THE HORIZONTAL REALIZATION OF $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$

The goal of this section is to introduce the “horizontal realization” of the algebra $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$. This allows to define the tensor product structure on the whole $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -category \mathcal{O} . It also provides a natural framework for the generalization of [FT1, Section 7] to K-theory/cohomology of $M(r, n)$. We prove that the natural vectors v_r^K, v_r^H in the appropriate completions of the modules M^r, V^r are eigenvectors with respect to a particular family of operators.

8.1. The horizontal realization via $\tilde{\mathcal{E}}$. We introduce a new realization of $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$.

Recall the distinguished collection of elements $\{u_{\mathbf{x}}, \kappa_{\mathbf{x}}\} \subset \ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ from Theorem 6.4. Note that there is a natural $\mathrm{SL}_2(\mathbb{Z})$ -action on $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)/(\psi_0^+ - 1) \simeq \tilde{\mathcal{E}}/(\kappa_{\mathbf{x}} - 1)_{\mathbf{x} \in \mathbb{Z}^2}$. In particular, we have a natural automorphism of $\tilde{\mathcal{E}}/(\kappa_{\mathbf{x}} - 1)_{\mathbf{x} \in \mathbb{Z}^2}$ induced by $u_{k,l} \mapsto u_{-l,k}$. Though there is no such automorphism for $\tilde{\mathcal{E}}/(\kappa_{0,1} - 1)$, we still have a nice presentation of this algebra in terms of the generators $\{u_{i,\pm 1}, u_{j,0}, \kappa_{1,0}\}$ rather than $\{u_{\pm 1,i}, u_{0,j}, \kappa_{1,0}\}$.

To formulate the main result, we need to introduce a modification of the algebra $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$, which we denote by $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$. The algebra $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$ is an associative unital \mathbb{C} -algebra generated by $\{\tilde{e}_i, \tilde{f}_i, \tilde{\psi}_j^{\pm}, \gamma^{\pm 1/2} | i \in \mathbb{Z}, j > 0\}$ with the following defining relations:

$$(TT0) \quad \tilde{\psi}^{\pm}(z) \tilde{\psi}^{\pm}(w) = \tilde{\psi}^{\pm}(w) \tilde{\psi}^{\pm}(z), \quad g(\gamma^{-1}w/z) \tilde{\psi}^+(z) \tilde{\psi}^-(w) = g(\gamma w/z) \tilde{\psi}^-(w) \tilde{\psi}^+(z),$$

$$(TT1) \quad \tilde{e}(z) \tilde{e}(w) = g(z/w) \tilde{e}(w) \tilde{e}(z),$$

$$(TT2) \quad \tilde{f}(z) \tilde{f}(w) = g(w/z) \tilde{f}(w) \tilde{f}(z),$$

$$(TT3) \quad (1 - q_1)(1 - q_2)(1 - q_3) \cdot [\tilde{e}(z), \tilde{f}(w)] = \delta(\gamma^{-1}z/w) \tilde{\psi}^+(\gamma^{1/2}w) - \delta(\gamma z/w) \tilde{\psi}^-(\gamma^{-1/2}w),$$

$$(TT4) \quad \tilde{\psi}^{\pm}(z) \tilde{e}(w) = g(\gamma^{\pm 1/2}z/w) \tilde{e}(w) \tilde{\psi}^{\pm}(z),$$

$$(TT5) \quad \tilde{\psi}^{\pm}(z) \tilde{f}(w) = g(\gamma^{\pm 1/2}w/z) \tilde{f}(w) \tilde{\psi}^{\pm}(z),$$

$$(TT6) \quad \mathrm{Sym}_{\mathfrak{S}_3}[\tilde{e}_{i_1}, [\tilde{e}_{i_2+1}, \tilde{e}_{i_3-1}]] = 0, \quad \mathrm{Sym}_{\mathfrak{S}_3}[\tilde{f}_{i_1}, [\tilde{f}_{i_2+1}, \tilde{f}_{i_3-1}]] = 0,$$

where $g(y) := \frac{(1-q_1y)(1-q_2y)(1-q_3y)}{(1-q_1^{-1}y)(1-q_2^{-1}y)(1-q_3^{-1}y)}$. Note that $g(y) = g(y^{-1})^{-1}$.

The following result is analogous to Theorem 6.4:

Theorem 8.1. *There is an isomorphism $\Xi_h : \tilde{\mathcal{E}}[\kappa_{1,0}^{\pm 1/2}]/(\kappa_{0,1} - 1) \xrightarrow{\sim} \ddot{\mathbf{U}}_{q_1, q_2, q_3}$ defined on the generators by*

$$\kappa_{1,0}^{\pm 1/2} \mapsto \gamma^{\pm 1/2}, \quad \theta_{\mp j,0} \mapsto \tilde{\psi}_j^{\pm}, \quad u_{-i,1} \mapsto \gamma^{|i|/2} \tilde{e}_i, \quad u_{-i,-1} \mapsto \gamma^{-|i|/2} \tilde{f}_i, \quad i \in \mathbb{Z}, j > 0.$$

Analogously to $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$, there is a similar coproduct Δ_h on the algebra $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$:

$$\Delta_h(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2}, \quad \Delta_h(\tilde{\psi}^{\pm}(z)) = \tilde{\psi}^{\pm}(\gamma_{(2)}^{\pm 1/2}z) \otimes \tilde{\psi}^{\pm}(\gamma_{(1)}^{\mp 1/2}z),$$

$$\Delta_h(\tilde{e}(z)) = \tilde{e}(z) \otimes 1 + \tilde{\psi}^-(\gamma_{(1)}^{1/2}z) \otimes \tilde{e}(\gamma_{(1)}z), \quad \Delta_h(\tilde{f}(z)) = 1 \otimes \tilde{f}(z) + \tilde{f}(\gamma_{(2)}z) \otimes \tilde{\psi}^+(\gamma_{(2)}^{1/2}z),$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2} \otimes 1$, $\gamma_{(2)}^{\pm 1/2} = 1 \otimes \gamma^{\pm 1/2}$ (see [DI]).

According to Theorems 6.4 and 8.1, the algebras $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)[(\psi_0^+)^{\pm 1/2}]$ and $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$ are isomorphic. In particular, we view Δ_h as a “horizontal coproduct” on the algebra $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$. It provides a tensor product structure on the category \mathcal{O} from Section 4.6. For two $\ddot{U}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -modules L_1, L_2 we denote the corresponding tensor product by $L_1 \otimes_h L_2$.

8.2. The horizontal realization of $F(u), V(u)$.

Let us describe the action of the currents $\tilde{e}(z), \tilde{f}(z), \tilde{\psi}^\pm(z)$ on the Fock module $F(u)$. Consider the Heisenberg Lie algebra \mathfrak{h} over \mathbb{C} with the generators $\{a_n\}_{n \in \mathbb{Z}}$ and the relations

$$[a_m, a_n] = m(1 - q_1^{|m|})/(1 - q_2^{-|m|})\delta_{m, -n}a_0.$$

Let $\mathfrak{h}^{\geq 0}$ be the subalgebra generated by $\{a_n\}_{n \geq 0}$ and $\mathcal{F} := \text{Ind}_{\mathfrak{h}^{\geq 0}}^{\mathfrak{h}} \mathbb{C}$ be the Fock \mathfrak{h} -representation.

Since the elements $\{\theta_{j,0}\} \subset \tilde{\mathcal{E}}$ form a Heisenberg Lie algebra and the highest weight vector $|\emptyset\rangle \in F(u)$ is annihilated by $\{\theta_{j,0}\}_{j < 0}$, we see that $F(u) \simeq \mathcal{F}$ as modules over the subalgebra generated by $\tilde{\psi}_j^\pm$. Together with the relations (TT4, TT5), we get the following result:

Proposition 8.2. *Identifying $F(u) \simeq \mathcal{F}$, the action of $\tilde{e}(z), \tilde{f}(z), \tilde{\psi}^\pm(z)$ is given by*

$$\begin{aligned} \rho_c(\gamma^{\pm 1/2}) &= q_3^{\pm 1/4}, \quad \rho_c(\tilde{\psi}^\pm(z)) = \exp\left(\mp \sum_{n>0} \frac{1 - q_2^{\mp n}}{n} (1 - q_3^n) q_3^{-n/4} a_{\pm n} z^{\mp n}\right), \\ \rho_c(\tilde{e}(z)) &= c \exp\left(\sum_{n>0} \frac{1 - q_2^n}{n} a_{-n} z^n\right) \exp\left(-\sum_{n>0} \frac{1 - q_2^{-n}}{n} a_n z^{-n}\right), \\ \rho_c(\tilde{f}(z)) &= c^{-1} \exp\left(-\sum_{n>0} \frac{1 - q_2^n}{n} q_3^{n/2} a_{-n} z^n\right) \exp\left(\sum_{n>0} \frac{1 - q_2^{-n}}{n} q_3^{n/2} a_n z^{-n}\right), \end{aligned}$$

where $c = (1 - q_3)u$.

These $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$ -representations $\{\rho_c\}$ were first considered in [FHHSY]. As we just explained, they correspond to the $\ddot{\mathbf{U}}'_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ -modules $\{F(u)\}$ under an identification of the two algebras.

Similarly one checks that action of currents $\tilde{e}(z), \tilde{f}(z), \tilde{\psi}^\pm(z)$ on the vector representation $V(u)$ coincides with the formulas for the $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$ -representations π_c considered in [FHHSY].

8.3. The matrix coefficient realization of \mathcal{A}^m .

We provide a new interpretation of \mathcal{A}^m .

For a $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$ -module L and two vectors $v_1, v_2 \in L$, we define

$$m_{v_1, v_2}(z_1, \dots, z_n) := \langle v_1 | \tilde{e}(z_1) \dots \tilde{e}(z_n) | v_2 \rangle \cdot \prod_{i < j} \omega^m(z_i, z_j) \in \mathbb{C}[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]].$$

The relation (TT1) implies that $m_{v_1, v_2}(z_1, \dots, z_n)$ is \mathfrak{S}_n -symmetric.

Proposition 8.3. *For $L = \rho_c$ and $v_1 = v_2 = \mathbf{1}$, we have*

$$m_{\mathbf{1}, \mathbf{1}}(z_1, \dots, z_n) = (-q_3)^{-n(n-1)/2} c^n \prod_{i < j} \frac{(z_i - q_3 z_j)(z_j - q_3 z_i)}{(z_i - z_j)^2}.$$

Proof.

For $n > 0$, we have $\exp(u \cdot a_n) \exp(v \cdot a_{-n}) = \exp(v \cdot a_{-n}) \exp(u \cdot a_n) \exp(uv \cdot n(1 - q_1^n)/(1 - q_2^{-n}))$. Therefore

$$\rho_c(\tilde{e}(z_i)) \rho_c(\tilde{e}(z_j)) = : \rho_c(\tilde{e}(z_i)) \rho_c(\tilde{e}(z_j)) : \cdot \prod_{n>0} \exp\left(-\frac{(1 - q_1^n)(1 - q_2^n)}{n} (z_j/z_i)^n\right).$$

It remains to use the equality $\prod_{n>0} \exp\left(-\frac{(1 - q_1^n)(1 - q_2^n)}{n} (z_j/z_i)^n\right) = \frac{(z_i - z_j)(z_i - q_1 q_2 z_j)}{(z_i - q_1 z_j)(z_i - q_2 z_j)}$. \square

In the case of $\rho_{c_1} \otimes_h \dots \otimes_h \rho_{c_m}$ we have the following result:

Proposition 8.4. *Consider $\bar{\mathbf{1}} := \mathbf{1} \otimes \dots \otimes \mathbf{1} \in \rho_{c_1} \otimes_h \dots \otimes_h \rho_{c_m}$. Then $m_{\bar{\mathbf{1}}, \bar{\mathbf{1}}}(z_1, \dots, z_n) \in \mathcal{A}^m$.*

Proof. Combining the formulas of Proposition 8.2 with formulas for Δ_h , we get

$$m_{\mathbf{1}, \mathbf{1}}(z_1, \dots, z_n) = \sum_f c_{f(1)} \cdots c_{f(n)} \prod_{i < j} \omega^m(z_i, z_j) \prod_{i < j} W_f(z_i, z_j),$$

where the sum is over all maps $f : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ and $W_f(z_i, z_j)$ is 1 (if $f(i) > f(j)$), is $\frac{(z_i - z_j)(z_i - q_1 q_2 z_j)}{(z_i - q_1 z_j)(z_i - q_2 z_j)}$ (if $f(i) = f(j)$) and is $g(z_i/z_j)$ (if $f(i) < f(j)$). The claim follows. \square

This realization of \mathcal{A}^m will be important in [FT2].

Remark 8.1. Same construction applied to $\pi_{c_1} \otimes_h \cdots \otimes_h \pi_{c_n}$ realizes the corresponding matrix coefficients as the classical Macdonald difference operators, see [FHHSY, Proposition A.10].

8.4. The Whittaker vector in the K-theory case. Let \widehat{M}^r be the completion of M^r with respect to a natural grading. Consider the *Whittaker vector* $v_r^K := \sum_{n \geq 0} [\mathcal{O}_{M(r,n)}] \in \widehat{M}^r$.

To state our main result, we introduce a family of the elements $\{K_i^{(m;j)}\}_{i > 0} \in S_i^m$ by

$$K_i^{(m;j)}(x_1, \dots, x_i) := K_i^m(x_1, \dots, x_i) x_1^j \cdots x_i^j = \prod_{a < b} \frac{(x_a - q_1 x_b)(x_b - q_1 x_a)}{(x_a - x_b)^2} \prod_s x_s^j.$$

Let $\{K_i^{(m;j)}\}_{i < 0}$ be analogous elements in the opposite algebra $(S^m)^{\text{opp}}$. The name “Whittaker” is motivated by the following result (see Appendix E for a proof):

Theorem 8.5. *The vector v_r^K is an eigenvector with respect to $\{K_{-n}^{(m;j)} | 0 \leq j \leq r, n > 0\}$. More precisely: $K_{-n}^{(m;j)}(v_r^K) = C_{j,-n} \cdot v_r^K$, where*

$$C_{0,-n} = (-1)^{n(n+1)/2 + nr - n} (t_1 t_2 \chi_1 \cdots \chi_r)^n \frac{t_1^{n(n-1)/2}}{(1-t_1)^n (1-t_2)(1-t_2^2) \cdots (1-t_2^n)},$$

$$C_{1,-n} = \cdots = C_{r-1,-n} = 0, \quad C_{r,-n} = \frac{(-t_1 t_2)^{n(n+1)/2}}{(1-t_1)^n (1-t_2)(1-t_2^2) \cdots (1-t_2^n)}.$$

Remark 8.2. Proposition 7.5 implies that the subalgebra of $(S^m)^{\text{opp}}$ generated by $\{K_{-n}^{(m;j)}\}_{0 \leq j \leq r}^{n \geq 0}$ corresponds to the subalgebra of $\check{U}_{q_1, q_2, q_3}^-(\mathfrak{gl}_1)$ generated by $\{f_j, [f_{j+1}, f_{j-1}], [f_{j+1}, [f_j, f_{j-1}]], \dots\}_{j=0}^r$.

8.5. The Whittaker vector in the cohomology case. Let \widehat{V}^r be the completion of V^r with respect to a natural grading. Consider the *Whittaker vector* $v_r^H := \sum_{n \geq 0} [M(r, n)] \in \widehat{V}^r$.

To state our main result, we introduce a family of the elements $\{K_i^{(a;j)}\}_{i > 0} \in S_i^a$ by

$$K_i^{(a;j)}(x_1, \dots, x_i) := K_i^a(x_1, \dots, x_i) x_1^j \cdots x_i^j = \prod_{a < b} \frac{(x_a - x_b - h_1)(x_b - x_a - h_1)}{(x_a - x_b)^2} \prod_s x_s^j.$$

Let $\{K_i^{(a;j)}\}_{i < 0}$ be analogous elements in the opposite algebra $(S^a)^{\text{opp}}$. The name “Whittaker” is motivated by the following result (see Appendix E for a proof):

Theorem 8.6. *The vector v_r^H is an eigenvector with respect to $\{K_{-n}^{(a;j)} | 0 \leq j \leq r, n > 0\}$. More precisely: $K_{-n}^{(a;j)}(v_r^H) = D_{j,-n} \cdot v_r^H$, where $D_{r,-n}$ is a degree n polynomial in x_a and*

$$D_{0,-n} = \cdots = D_{r-2,-n} = 0, \quad D_{r-1,-n} = \frac{(-1)^{n(n+1)/2 + nr - n}}{n! s_1^n s_2^n}, \quad D_{r,-1} = \frac{(-1)^{r+1}}{s_1 s_2} \sum_{a=1}^r x_a.$$

Remark 8.3. Proposition 7.8 implies that the subalgebra of $(S^a)^{\text{opp}}$ generated by $\{K_{-n}^{(a;j)}\}_{0 \leq j \leq r}^{n \geq 0}$ corresponds to the subalgebra of $\check{Y}_{h_1, h_2, h_3}^-(\mathfrak{gl}_1)$ generated by $\{f_j, [f_j, f_{j+1}], [f_j, [f_j, f_{j+2}]], \dots\}_{j=0}^r$.

APPENDIX A. THE TRIANGULAR DECOMPOSITION

Let $\ddot{U}^-, \ddot{U}^0, \ddot{U}^+$ be the subalgebras of $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ generated by $\{f_i\}$, $\{\psi_j^\pm, (\psi_0^\pm)^{-1}\}$, $\{e_i\}$.

Proposition A.1. (a) (Triangular decomposition for $\ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$) The multiplication map $m : \ddot{U}^- \otimes \ddot{U}^0 \otimes \ddot{U}^+ \rightarrow \ddot{U}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ is an isomorphism of vector spaces.
 (b) The subalgebras $\ddot{U}^-, \ddot{U}^+, \ddot{U}^0$ are generated by $\{f_i\}$, $\{e_i\}$, $\{\psi_j^\pm, (\psi_0^\pm)^{-1}\}$ with the defining relations (T2, T6), (T1, T6), and (T0), respectively.

Remark A.1. The same results also hold for $\ddot{Y}_{h_1, h_2, h_3}(\mathfrak{gl}_1)$ in a completely analogous way.

The proof is standard. Consider an associative algebra $\ddot{V}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ generated by $e_i, f_i, \psi_j^\pm, (\psi_0^\pm)^{-1}$ subject to the relations (T0, T3, T4, T5). We define the subalgebras $\ddot{V}^-, \ddot{V}^0, \ddot{V}^+$ of $\ddot{V}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ in the same way. Let I^\pm be the two-sided ideal of $\ddot{V}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ generated by the quadratic and cubic relations in e_i and f_i arising from (T1, T2, T6). Explicitly, I^+ is generated by

$$A_{i,j} = e_{i+3}e_j - \sigma_1 e_{i+2}e_{j+1} + \sigma_2 e_{i+1}e_{j+2} - e_i e_{j+3} - e_j e_{i+3} + \sigma_2 e_{j+1}e_{i+2} - \sigma_1 e_{j+2}e_{i+1} + e_{j+3}e_i,$$

$$B_{i_1, i_2, i_3} = \text{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2+1}, e_{i_3-1}]].$$

We also let J^\pm stay for the corresponding two-sided ideals of \ddot{V}^\pm . Proposition A.1 follows from:

Lemma A.2. (a) (Triangular decomposition for $\ddot{V}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$) The multiplication map $m : \ddot{V}^- \otimes \ddot{V}^0 \otimes \ddot{V}^+ \rightarrow \ddot{V}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$ is an isomorphism of vector spaces.
 (b) The subalgebras \ddot{V}^-, \ddot{V}^+ are free associative algebra in $\{f_i\}$, $\{e_i\}$, respectively. The subalgebra \ddot{V}^0 is generated by $\psi_j^\pm, (\psi_0^\pm)^{-1}$ with the defining relations (T0).
 (c) We have $I^+ = m(\ddot{V}^- \otimes \ddot{V}^0 \otimes J^+)$ and $I^- = m(J^- \otimes \ddot{V}^0 \otimes \ddot{V}^+)$.

Proof of Lemma A.2.

Part (a) is standard. Part (b) follows immediately from (a).

Part (c) is equivalent to $\ddot{V}^- \ddot{V}^0 J^+$ being a two-sided ideal of $\ddot{V}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$. Using the equality $\ddot{V}^- \ddot{V}^0 \ddot{V}^+ = \ddot{V}_{q_1, q_2, q_3}(\mathfrak{gl}_1)$, we reduce to showing $[A_{i,j}, t_r], [B_{i_1, i_2, i_3}, t_r], [A_{i,j}, f_r], [B_{i_1, i_2, i_3}, f_r] \in \ddot{V}^0 J^+$. Relation (T4t) implies that the first two commutators are just the linear combinations of $A_{i', j'}$ and $B_{i'_1, i'_2, i'_3}$. Also $[A_{i,j}, f_r] = 0$ (it is a sum of two quadratic expressions from (T4)).

To prove $[B_{i_1, i_2, i_3}, f_r] \in \ddot{V}^0 J^+$ we work with the generating series. The relation (T3) implies

$$\beta_1 \cdot [e(z_1)e(z_2)e(z_3), f(w)] =$$

$$\delta\left(\frac{z_1}{w}\right)\psi(z_1)e(z_2)e(z_3) + \delta\left(\frac{z_2}{w}\right)\psi(z_2)e(z_1)e(z_3)\rho(z_2, z_1) + \delta\left(\frac{z_3}{w}\right)\psi(z_3)e(z_1)e(z_2)\rho(z_3, z_1)\rho(z_3, z_2).$$

where $\rho(x, y) := g(y/x) = -\frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(y-q_1x)(y-q_2x)(y-q_3x)}$ and $\psi(z) = \psi^+(z) - \psi^-(z)$. Hence, we have

$$\left[\text{Sym}_{\mathfrak{S}_3} \left\{ \left(\frac{z_2}{z_1} + \frac{z_2}{z_3} - \frac{z_1}{z_2} - \frac{z_3}{z_2} \right) e(z_1)e(z_2)e(z_3) \right\}, f(w) \right] =$$

$$\beta_1^{-1} (\delta(z_1/w)\psi(z_1)C_1(z_2, z_3) + \delta(z_2/w)\psi(z_2)C_2(z_3, z_1) + \delta(z_3/w)\psi(z_3)C_3(z_1, z_2)),$$

where $C_1(z_2, z_3) = e(z_2)e(z_3)C_{123} + e(z_3)e(z_2)C_{132}$ and

$$C_{123} = \left(\frac{z_2}{z_1} + \frac{z_2}{z_3} - \frac{z_1}{z_2} - \frac{z_3}{z_2} \right) + \rho(z_1, z_2) \left(\frac{z_1}{z_2} + \frac{z_1}{z_3} - \frac{z_2}{z_1} - \frac{z_3}{z_1} \right) + \rho(z_1, z_2)\rho(z_1, z_3) \left(\frac{z_3}{z_1} + \frac{z_3}{z_2} - \frac{z_1}{z_3} - \frac{z_2}{z_3} \right),$$

$$C_{132} = \left(\frac{z_3}{z_1} + \frac{z_3}{z_2} - \frac{z_1}{z_3} - \frac{z_2}{z_3} \right) + \rho(z_1, z_3) \left(\frac{z_1}{z_2} + \frac{z_1}{z_3} - \frac{z_2}{z_1} - \frac{z_3}{z_1} \right) + \rho(z_1, z_2)\rho(z_1, z_3) \left(\frac{z_2}{z_1} + \frac{z_2}{z_3} - \frac{z_1}{z_2} - \frac{z_3}{z_2} \right).$$

The equality $C_{132} = -\rho(z_3, z_2)C_{123}$ implies actually that $C_1(z_2, z_3)$ is proportional to the generating function of $A_{i,j}$. The same results apply to $C_2(z_3, z_1), C_3(z_1, z_2)$. This yields the inclusion $[B_{i_1, i_2, i_3}, f_r] \in \ddot{V}^0 J^+$ for any $i_1, i_2, i_3, r \in \mathbb{Z}$. \square

APPENDIX B. PROOFS OF THEOREMS 3.2, 3.4

B.1. Sketch of the proof of Theorem 3.2. The purpose of this section is to outline the main computation required to carry out verifications of (T0)-(T6) in the proof of Theorem 3.2.

The verification of relations (T0, T1, T2, T6t) is straightforward just by using the formulas for the matrix coefficients from Lemma 3.3(a). It is also easy to see that the operators $[e_i, f_j]$ are diagonalizable in the fixed point basis and depend on $i + j$ only: $[e_i, f_j]([\bar{\lambda}]) = \gamma_{i+j|\bar{\lambda}} \cdot [\bar{\lambda}]$.

Next, we introduce series of operators $\phi^\pm(z) = \sum_{i=0}^\infty \phi_i^\pm z^{\mp i}$, diagonalizable in the fixed point basis and satisfying the equation

$$[e(z), f(w)] = \frac{\delta(z/w)}{(1-t_1)(1-t_2)(1-t_3)} (\phi^+(w) - \phi^-(z)).$$

Actually, this determines $\phi_{>0}^\pm$ and $\phi_0^+ - \phi_0^-$ uniquely. Our next goal is to specify ϕ_0^\pm .

Lemma B.1. *We have*

$$\begin{aligned} \gamma_{0|\bar{\lambda}} &= (-1)^{r-1} \chi_1 \cdots \chi_r \frac{t_1 t_2 - t_1^{r+1} t_2^{r+1}}{(1-t_1)(1-t_2)(1-t_3)}, \\ \gamma_{1|\bar{\lambda}} &= (-1)^r \chi_1 \cdots \chi_r t_1^{r+1} t_2^{r+1} \left(\sum_{a=1}^r \chi_a^{-1} - \sum_{a=1}^r \sum_{\square \in \lambda^a} \chi(\square) \right). \end{aligned}$$

Proof.

Fix positive integers $L_a > \lambda_1^{a*}$. Applying Lemma 3.3(a), we find:

(\heartsuit)

$$\begin{aligned} \gamma_{s|\bar{\lambda}} &= \sum_{l=1}^r \sum_{j=1}^{L_l} \frac{t_1^2 t_2^2}{(1-t_1)^2} (\chi_j^{(l)})^{s-r} \cdot \frac{\chi_j^{(l)} (1 - \chi_j^{(l)} t_2^{1-L_l} t_1 \chi_l)}{\chi_j^{(l)} - t_2^{L_l} \chi_l^{-1}} \cdot \prod_{k \neq j}^{k \leq L_l} \frac{(\chi_j^{(l)} - t_1 t_2 \chi_k^{(l)}) (\chi_k^{(l)} - t_2 \chi_j^{(l)})}{(\chi_j^{(l)} - t_1 \chi_k^{(l)}) (\chi_k^{(l)} - \chi_j^{(l)})} \times \\ &\quad \prod_{a \neq l} \left(\frac{\chi_j^{(l)} (1 - \chi_j^{(l)} t_2^{1-L_a} t_1 \chi_a)}{\chi_j^{(l)} - t_2^{L_a} \chi_a^{-1}} \cdot \prod_{k=1}^{L_a} \frac{(\chi_j^{(l)} - t_1 t_2 \chi_k^{(a)}) (\chi_k^{(a)} - t_2 \chi_j^{(l)})}{(\chi_j^{(l)} - t_1 \chi_k^{(a)}) (\chi_k^{(a)} - \chi_j^{(l)})} \right) - \\ &\quad \sum_{l=1}^r \sum_{j=1}^{L_l} \frac{t_1^2 t_2^2}{(1-t_1)^2} (t_1 \chi_j^{(l)})^{s-r} \cdot \frac{\chi_j^{(l)} (1 - \chi_j^{(l)} t_2^{1-L_l} t_1^2 \chi_l)}{\chi_j^{(l)} - t_2^{L_l} t_1^{-1} \chi_l^{-1}} \cdot \prod_{k \neq j}^{k \leq L_l} \frac{(\chi_k^{(l)} - t_1 t_2 \chi_j^{(l)}) (\chi_j^{(l)} - t_2 \chi_k^{(l)})}{(\chi_k^{(l)} - t_1 \chi_j^{(l)}) (\chi_j^{(l)} - \chi_k^{(l)})} \times \\ &\quad \prod_{a \neq l} \left(\frac{\chi_j^{(l)} (1 - \chi_j^{(l)} t_2^{1-L_a} t_1^2 \chi_a)}{\chi_j^{(l)} - t_2^{L_a} t_1^{-1} \chi_a^{-1}} \cdot \prod_{k=1}^{L_a} \frac{(\chi_k^{(a)} - t_1 t_2 \chi_j^{(l)}) (\chi_j^{(l)} - t_2 \chi_k^{(a)})}{(\chi_k^{(a)} - t_1 \chi_j^{(l)}) (\chi_j^{(l)} - \chi_k^{(a)})} \right), \end{aligned}$$

where $\chi_k^{(a)} = t_1^{\lambda_k^{a*}-1} t_2^{k-1} \chi_a^{-1}$ as before. The result does not depend on the choice of $\{L_a\}$.

(i) For $s = 0$, the right hand side of (\heartsuit) is a degree 0 rational function in the variables $\chi_k^{(a)}$. It is easy to see that it has no poles, in fact. Therefore, it is an element of $\mathbb{C}(t_1, t_2, \chi_1, \dots, \chi_r)$ independent of $\bar{\lambda}$. It suffices to compute its value at the empty r -partition \emptyset . For $\bar{\lambda} = \emptyset$, we can choose $L_1 = \dots = L_r = 1$, while $\chi_k^{(a)} = t_1^{-1} t_2^{k-1} \chi_a^{-1}$ for $k \geq 1$. Applying (\heartsuit), we get

$$\begin{aligned} \gamma_{0|\bar{\lambda}} &= \gamma_{0|\emptyset} = -\frac{t_1^2 t_2^2}{(1-t_1)^2} \sum_{l=1}^r t_1^{-r} \frac{1-t_1}{t_1^{-1} \chi_l^{-1} - t_2 t_1^{-1} \chi_l^{-1}} \prod_{a \neq l} \frac{(t_1^{-1} \chi_a^{-1} - t_2 \chi_l^{-1})(1 - t_1 \chi_a \chi_l^{-1})}{(t_1^{-1} \chi_l^{-1} - t_1^{-1} \chi_a^{-1})(t_1^{-1} \chi_a^{-1} - \chi_l^{-1})} = \\ &= \frac{(-1)^r t_1^2 t_2^2}{(1-t_1)(1-t_2)} \chi_1 \cdots \chi_r \sum_{l=1}^r \prod_{a \neq l} \frac{\chi_l - t_1 t_2 \chi_a}{\chi_l - \chi_a} = \frac{(-1)^r t_1^2 t_2^2}{(1-t_1)(1-t_2)} \chi_1 \cdots \chi_r \frac{1-t_1 t_2^r}{1-t_1 t_2}, \end{aligned}$$

where we used the identity $\sum_{l=1}^r \prod_{a \neq l} \frac{\chi_l - u \chi_a}{\chi_l - \chi_a} = \frac{1-u^r}{1-u}$. The first result follows.

(ii) For $s = 1$, the right hand side of (\heartsuit) is a degree 1 rational function in the variables $\chi_k^{(a)}$. It is easy to see that it has no poles, actually. Therefore, it is a linear function. Its leading term equals $(-1)^r \chi_1 \dots \chi_r \frac{t_1^{r+2} t_2^{r+1}}{1-t_1} \cdot \sum_{l=1}^r \sum_{j=1}^{L_l} \chi_j^{(l)}$. Hence, we have:

$$\gamma_{1|\bar{\lambda}} = (-1)^r \chi_1 \dots \chi_r \frac{t_1^{r+2} t_2^{r+1}}{1-t_1} \cdot \sum_{l=1}^r \sum_{j=1}^{\infty} \tilde{\chi}_j^{(l)} + C$$

for a constant $C \in \mathbb{C}(t_1, t_2, \chi_1, \dots, \chi_r)$ independent of $\bar{\lambda}$, where $\tilde{\chi}_j^{(l)} := \chi_j^{(l)} - t_1^{-1} t_2^{j-1} \chi_a^{-1}$.

Note that $\sum_{a=1}^r \sum_{\square \in \lambda^a} \chi(\square) = \sum_{a=1}^r \sum_{j=1}^{\lambda_1^a} t_2^{j-1} (1+t_1+\dots+t_1^{\lambda_j^a-1}) \chi_a^{-1} = \sum_{l=1}^r \sum_j \frac{-t_1}{1-t_1} \tilde{\chi}_j^{(l)}$. On the other hand, $C = \gamma_{1|\emptyset}$. Applying (\heartsuit) , we get

$$\begin{aligned} C = \gamma_{1|\emptyset} &= -\frac{t_1^2 t_2^2}{(1-t_1)^2} \sum_{l=1}^r t_1^{-r} \frac{\chi_l^{-1} (1-t_1)}{t_1^{-1} \chi_l^{-1} - t_2 t_1^{-1} \chi_l^{-1}} \prod_{a \neq l} \frac{(t_1^{-1} \chi_a^{-1} - t_2 \chi_l^{-1})(1-t_1 \chi_a \chi_l^{-1})}{(t_1^{-1} \chi_l^{-1} - t_1^{-1} \chi_a^{-1})(t_1^{-1} \chi_a^{-1} - \chi_l^{-1})} = \\ &= \frac{(-1)^r t_1^2 t_2^2}{(1-t_1)(1-t_2)} \chi_1 \dots \chi_r \sum_{l=1}^r \frac{1}{\chi_l} \prod_{a \neq l} \frac{\chi_l - t_1 t_2 \chi_a}{\chi_l - \chi_a} = \frac{(-1)^r t_1^{r+1} t_2^{r+1}}{(1-t_1)(1-t_2)} \chi_1 \dots \chi_r \sum_{l=1}^r \chi_l^{-1}, \end{aligned}$$

where we used $\sum_{l=1}^r \frac{1}{\chi_l} \prod_{a \neq l} \frac{\chi_l - u \chi_a}{\chi_l - \chi_a} = u^{r-1} \sum_{l=1}^r \chi_l^{-1}$. The second result follows. \square

Due to the first equality of this lemma, we can set

$$\phi_0^+ := (-1)^r t_1^{r+1} t_2^{r+1} \chi_1 \dots \chi_r, \quad \phi_0^- := (-1)^r t_1 t_2 \chi_1 \dots \chi_r.$$

Next, we claim that $\phi^\pm(z)$ satisfy the following relations:

$$(10) \quad \phi^\pm(z) e(w) (z - q_1 w) (z - q_2 w) (z - q_3 w) = -e(w) \phi^\pm(z) (w - q_1 z) (w - q_2 z) (w - q_3 z)$$

$$(11) \quad \phi^\pm(z) f(w) (w - q_1 z) (w - q_2 z) (w - q_3 z) = -f(w) \phi^\pm(z) (z - q_1 w) (z - q_2 w) (z - q_3 w)$$

The proof is based on straightforward computations in the fixed point basis.

Finally, relation (10) implies the following identity:

$$\phi^+(z)_{|\bar{\lambda} + \square_j^l} = \phi^+(z)_{|\bar{\lambda}} \cdot \frac{(1 - t_1^{-1} \chi(\square_j^l)/z)(1 - t_2^{-1} \chi(\square_j^l)/z)(1 - t_3^{-1} \chi(\square_j^l)/z)}{(1 - t_1 \chi(\square_j^l)/z)(1 - t_2 \chi(\square_j^l)/z)(1 - t_3 \chi(\square_j^l)/z)}.$$

Therefore,

$$\phi^+(z)_{|\bar{\lambda}} = \phi^+(z)_{|\emptyset} \cdot c_r(z)_{|\bar{\lambda}}^+.$$

Applying (\heartsuit) once again, we get:

$$\phi^+(z)_{|\emptyset} = (\phi_0^- + \sum_{i \geq 0} (1-t_1)(1-t_2)(1-t_3) \gamma_i z^{-i})_{|\emptyset} = (-1)^r t_1 t_2 \chi_1 \dots \chi_r +$$

$$t_1 t_2 (1-t_1 t_2) \chi_1 \dots \chi_r \sum_{l=1}^r \frac{1}{1 - \chi_l^{-1} z^{-1}} \prod_{a \neq l} \frac{\chi_l - t_1 t_2 \chi_a}{\chi_a - \chi_l} = (-1)^r t_1 t_2 \chi_1 \dots \chi_r \prod_{l=1}^r \left(\frac{1 - t_1 t_2 \chi_l z}{1 - \chi_l z} \right)^+,$$

where we used the identity

$$1 - (1-u) \sum_{l=1}^r \frac{1}{1 - 1/(\chi_l z)} \prod_{a \neq l} \frac{\chi_l - u \chi_a}{\chi_l - \chi_a} = \prod_{l=1}^r \frac{u - \frac{1}{\chi_l z}}{1 - \frac{1}{\chi_l z}}.$$

This proves $\phi^+(z) = \psi^+(z)$. The same arguments imply $\phi^-(z) = \psi^-(z)$. The relation (T3) follows. On the other hand, relations (10) and (11) imply that the relations (T4, T5) also hold.

This completes the proof of Theorem 3.2.

B.2. Sketch of the proof of Theorem 3.4. The proof of the cohomological counterpart of the previous result is completely analogous and is parallel to the proof of Lemma 2.4.

The verification of the relations (Y0, Y1, Y2, Y6) is straightforward. To verify the remaining relations, we follow the same pattern as above. It is easy to check that $[e_i, f_j]$ is diagonalizable in the fixed point basis and depends on $i + j$ only: $[e_i, f_j](\bar{\lambda}) = \gamma_{i+j|\bar{\lambda}} \cdot \bar{\lambda}$.

Lemma B.2. *We have $\gamma_{0|\bar{\lambda}} = \frac{-r}{s_1 s_2}$, $\gamma_{1|\bar{\lambda}} = \frac{1}{s_1 s_2} \left(\sum_{a=1}^r x_a - \binom{r}{2} (s_1 + s_2) \right)$,*

$$\gamma_{2|\bar{\lambda}} = 2|\bar{\lambda}| - \frac{1}{s_1 s_2} \left(\sum_{a=1}^r x_a^2 - (r-1)(s_1 + s_2) \sum_{a=1}^r x_a + \binom{r}{3} (s_1 + s_2)^2 \right).$$

Proof.

Applying Lemma 3.5(a), we find:

(♠)

$$\begin{aligned} \gamma_{s|\bar{\lambda}} = & \sum_{l=1}^r \sum_{j=1}^{L_l} \frac{1}{s_1^2} (x_j^{(l)})^s \cdot \frac{x_j^{(l)} + (1-L_l)s_2 + s_1 + x_l}{-x_j^{(l)} + L_l s_2 - x_l} \cdot \prod_{k \neq j}^{k \leq L_l} \frac{(x_j^{(l)} - x_k^{(l)} - s_1 - s_2)(x_k^{(l)} - x_j^{(l)} - s_2)}{(x_j^{(l)} - x_k^{(l)} - s_1)(x_k^{(l)} - x_j^{(l)})} \times \\ & \prod_{a \neq l} \left(\frac{x_j^{(l)} + (1-L_a)s_2 + s_1 + x_a}{x_j^{(l)} - L_a s_2 + x_a} \cdot \prod_{k=1}^{L_a} \frac{(x_j^{(l)} - x_k^{(a)} - s_1 - s_2)(x_k^{(a)} - x_j^{(l)} - s_2)}{(x_j^{(l)} - x_k^{(a)} - s_1)(x_k^{(a)} - x_j^{(l)})} \right) - \\ & \sum_{l=1}^r \sum_{j=1}^{L_l} \frac{1}{s_1^2} (x_j^{(l)} + s_1)^s \cdot \frac{x_j^{(l)} + (1-L_l)s_2 + 2s_1 + x_l}{-x_j^{(l)} + L_l s_2 - s_1 - x_l} \cdot \prod_{k \neq j}^{k \leq L_l} \frac{(x_k^{(l)} - x_j^{(l)} - s_1 - s_2)(x_j^{(l)} - x_k^{(l)} - s_2)}{(x_k^{(l)} - x_j^{(l)} - s_1)(x_j^{(l)} - x_k^{(l)})} \times \\ & \prod_{a \neq l} \left(\frac{x_j^{(l)} + (1-L_a)s_2 + 2s_1 + x_a}{x_j^{(l)} - L_a s_2 + s_1 + x_a} \cdot \prod_{k=1}^{L_a} \frac{(x_k^{(a)} - x_j^{(l)} - s_1 - s_2)(x_j^{(l)} - x_k^{(a)} - s_2)}{(x_k^{(a)} - x_j^{(l)} - s_1)(x_j^{(l)} - x_k^{(a)})} \right), \end{aligned}$$

where $x_k^{(a)} = (\lambda_k^a - 1)s_1 + (k-1)s_2 - x_a$ as before. The right hand side of (♠) is a degree s rational function in the variables $x_k^{(a)}$. Actually, it is easy to see that it has no poles for $s \geq 0$.

(i) For $s = 0$, we therefore get an element of $\mathbb{C}(s_1, s_2, x_1, \dots, x_l)$ independent of $\bar{\lambda}$. Using (♠) once again, we get $\gamma_{0|\bar{\lambda}} = \gamma_{0|\bar{0}} = -r/s_1 s_2$.

(ii) For $s = 1$, we therefore get a linear function. But its leading term is zero, in fact. So $\gamma_{1|\bar{\lambda}} = \gamma_{1|\bar{0}}$. Using (♠) once again, we get $\gamma_{1|\bar{\lambda}} = \gamma_{1|\bar{0}} = \frac{1}{s_1 s_2} \left(\sum_{a=1}^r x_a - \binom{r}{2} (s_1 + s_2) \right)$.

(iii) For $s = 2$, we therefore get a quadratic function. But its leading quadratic part is zero, in fact. So $\gamma_{2|\bar{\lambda}}$ is a linear function. Similarly to the proof of Lemma 2.4, we find that the leading linear part is actually $\frac{2}{s_1} \sum_{a=1}^r \sum_{k=1}^\infty \tilde{x}_k^{(a)} = 2|\bar{\lambda}|$, where $\tilde{x}_k^{(a)} := x_k^{(a)} - (-s_1 + (k-1)s_2 - x_a)$. Hence, $\gamma_{2|\bar{\lambda}} = 2|\bar{\lambda}| + \gamma_{2|\bar{0}}$. Applying (♠) once again, we get the last formula. \square

Using this lemma together with computations in the fixed point basis, it is straightforward to check that $\{\phi_i, e_i, f_i\}_{i \in \mathbb{Z}_+}$ satisfy the relations (Y4, Y4', Y5, Y5'). This in turn implies

$$\phi(z)_{|\bar{\lambda} + \square_j^l} = \phi(z)_{|\bar{\lambda}} \cdot \frac{(z - \chi(\square_j^l) + s_1)(z - \chi(\square_j^l) + s_2)(z - \chi(\square_j^l) + s_3)}{(z - \chi(\square_j^l) - s_1)(z - \chi(\square_j^l) - s_2)(z - \chi(\square_j^l) - s_3)},$$

where $\phi(z) := 1 + \sigma_3 \sum_{i \geq 0} \phi_i z^{-i-1}$. Therefore, $\phi(z)_{|\bar{\lambda}} = \phi(z)_{|\bar{0}} \cdot C_r(z)_{|\bar{\lambda}}^+$. Applying (♠), we get:

$$\phi(z)_{|\bar{0}} = 1 - \frac{\sigma_3}{s_1 s_2} \sum_{i \geq 0} \sum_{l=1}^r (-x_l)^i z^{-i-1} \prod_{a \neq l} \frac{x_l - x_a - s_1 - s_2}{x_l - x_a} = 1 - s_3 \sum_{l=1}^r \frac{1}{z + x_l} \prod_{a \neq l} \frac{x_l - x_a - s_1 - s_2}{x_l - x_a}.$$

Combining with the identity $1 + u \sum_{l=1}^r \frac{1}{z + x_l} \prod_{a \neq l} \frac{x_l - x_a - u}{x_l - x_a} = \prod_{j=1}^r \frac{z + x_j + u}{z + x_j}$, we get $\phi(z) = \psi(z)$.

APPENDIX C. PROOFS OF THEOREMS 5.5, 5.9

C.1. Proof of Theorem 5.5. We prove that θ_m is an isomorphism of \mathbb{C} -algebras for $h_0 \notin \mathbb{Q} \cdot \pi i$.

As mentioned in Section 5.4, all the defining relations of the algebra $\ddot{U}'_{h_0}(\mathfrak{gl}_1)$ are of Lie-type. Therefore, it is the universal enveloping algebra of the Lie algebra \ddot{u}'_{h_0} generated by $\{e_i, f_i, H_m, \kappa\}$ with the same defining relations. Moreover, \ddot{u}'_{h_0} is a $\mathbb{C} \cdot \kappa$ -central extension of the Lie-algebra \ddot{u}_{h_0} generated by $\{e_i, f_i, H_m\}$ with the following defining relations:

$$\begin{aligned}
(\text{u0}) \quad & [H_k, H_l] = 0, \\
(\text{u1}) \quad & [e_{i+3}, e_j] - (1 + q + q^{-1})[e_{i+2}, e_{j+1}] + (1 + q + q^{-1})[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] = 0, \\
(\text{u2}) \quad & [f_{i+3}, f_j] - (1 + q + q^{-1})[f_{i+2}, f_{j+1}] + (1 + q + q^{-1})[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] = 0, \\
(\text{u3}) \quad & [e_i, f_j] = H_{i+j} \text{ for } j \neq -i, \quad [e_i, f_{-i}] = 0, \\
(\text{u4}) \quad & [H_m, e_i] = -(1 - q^m)(1 - q^{-m})e_{i+m}, \\
(\text{u5}) \quad & [H_m, f_i] = (1 - q^m)(1 - q^{-m})f_{i+m}, \\
(\text{u6}) \quad & [e_0, [e_1, e_{-1}]] = 0, \quad [f_0, [f_1, f_{-1}]] = 0,
\end{aligned}$$

where $q := e^{h_0} \in \mathbb{C}^*$. Note that $h_0 \notin \mathbb{Q} \cdot \pi i$ iff $q \neq \sqrt{1}$ (not a root of 1).

Hence, it suffices to check that the corresponding homomorphism $\theta_m : \ddot{u}_{h_0} \rightarrow \mathfrak{d}'_{h_0}$ defined by

$$\theta_m : e_i \mapsto Z^i D, \quad f_i \mapsto -D^{-1} Z^i, \quad H_k \mapsto (q^{-k} - 1)Z^k$$

is an isomorphism of the \mathbb{C} -Lie algebras for $q \neq \sqrt{1}$. The surjectivity of θ_m is clear.

The Lie algebra \ddot{u}_{h_0} is \mathbb{Z}^2 -graded, where we set $\deg(e_i) = (i, 1)$, $\deg(f_i) = (i, -1)$, $\deg(H_k) = (k, 0)$. The Lie algebra \mathfrak{d}'_{h_0} is also \mathbb{Z}^2 -graded, where we set $\deg(Z^i D^j) = (i, j)$. Moreover, θ_m intertwines those \mathbb{Z}^2 -gradings. Since $\dim(\mathfrak{d}'_{h_0})_{i,j} = 1$ for $(i, j) \neq (0, 0)$, it suffices to show that $\dim(\ddot{u}_{h_0})_{i,j} \leq 1$. This statement is clear for $j = 0$. In the remaining part we prove it for $j > 0$.

Let $\ddot{u}_{h_0}^{\geq 0}$ be the Lie algebra, generated by $\{e_i, H_m\}$ with the defining relations (u0, u1, u4, u6). It suffices to prove that $\dim(\ddot{u}_{h_0}^{\geq 0})_{i,j} \leq 1$ for $j > 0$. We prove this by an induction on j .

- *Case $j = 1$.*

It is clear that $(\ddot{u}_{h_0}^{\geq 0})_{N,1}$ is spanned by e_N .

- *Case $j = 2$.*

It is clear that $(\ddot{u}_{h_0}^{\geq 0})_{N,2}$ is spanned by $\{[e_{i_1}, e_{i_2}]\}_{i_1+i_2=N}$. However, (u1) implies

$$[e_{i+2+k}, e_{i+1-k}] = \frac{q^{k+1} - q^{-k}}{q - 1} [e_{i+2}, e_{i+1}], \quad [e_{i+2+k}, e_{i-k}] = \frac{q^{k+2} - q^{-k}}{q^2 - 1} [e_{i+2}, e_i].$$

These formulas can be unified in the following way:

$$(12) \quad [e_{i_1}, e_{i_2}] = \frac{q^{i_2} - q^{i_1}}{q^{i_1+i_2} - 1} [e_0, e_{i_1+i_2}] \text{ if } i_1 + i_2 \neq 0, \quad [e_i, e_{-i}] = \frac{q^{i+1} - q^{1-i}}{q^2 - 1} [e_1, e_{-1}].$$

Therefore, we see that $(\ddot{u}_{h_0}^{\geq 0})_{N,2}$ is either spanned by $[e_0, e_N]$ (if $N \neq 0$) or $[e_1, e_{-1}]$ (if $N = 0$).

- *Case $j = 3$.*

Let us introduce the following common notation: $[a_1; a_2; \dots; a_n]_n := [a_1, [a_2, [\dots [a_{n-1}, a_n]]]]$. The space $(\ddot{u}_{h_0}^{\geq 0})_{N,3}$ is spanned by $\{[e_{i_1}; e_{i_2}; e_{i_3}]\}_{i_1+i_2+i_3=N}$. Using the automorphism π of the Lie algebra $\ddot{u}_{h_0}^{\geq 0}$, defined by $e_i \mapsto e_{i+1}$, $H_m \mapsto H_m$, we can assume $i_1, i_2, i_3 > 0$. Together with the case $j = 2$, it suffices to show that $[e_k; e_0; e_l]$ is a multiple of $[e_0; e_0; e_{k+l}]$ for any $k, l > 0$.

Define $\mathbf{h}_m := -\frac{mH_m}{(1-q^m)(1-q^{-m})}$ for $m \neq 0$. Then $\text{ad}(\mathbf{h}_m)(e_i) = e_{i+m}$. Therefore:

$$\text{ad}(\mathbf{h}_1)([e_k; e_0; e_l]) = [e_{k+1}; e_0; e_l] + [e_k; e_1; e_l] + [e_k; e_0; e_{l+1}].$$

Assuming $[e_k; e_0; e_l]$ is a multiple of $[e_0; e_0; e_{k+l}]$, we get $[e_{k+1}; e_0; e_l]$ is a linear combination of $[e_0; e_0; e_{k+l+1}]$ and $[e_1; e_0; e_{k+l}]$ (we use (12) there). It remains to consider $k = 1$ case.

We will prove by an induction on $N > 1$ that $[e_1; e_0; e_{N-1}] = \frac{(q^{N-1}-q^2)(q^{N-1}-1)}{(q^{N-1}-1)^2}[e_0; e_0; e_N]$. This is equivalent to $[e_1; e_0; e_{N-1}]$ being a multiple of $[e_0; e_0; e_N]$, since we can recover the constant $\lambda_{N,3} := \frac{(q^{N-1}-q^2)(q^{N-1}-1)}{(q^{N-1}-1)^2}$ by comparing the images $\theta_m([e_1; e_0; e_{N-1}])$ and $\theta_m([e_0; e_0; e_N])$.
 ◦ Case $N = 2$.

Recall that the relation (u6) combined with (u4) imply

$$(u6') \quad \text{Sym}[e_{i_1}; e_{i_2+1}; e_{i_3-1}] = 0 \quad \forall i_1, i_2, i_3 \in \mathbb{Z}.$$

Plugging $i_1 = 1, i_2 = 1, i_3 = 0$, we get $[e_1; e_2; e_{-1}] + [e_1; e_1; e_0] + [e_0; e_2; e_0] = 0$. Combining this equality with (12), we get $[e_0; e_0; e_2] = -\frac{(q+1)^2}{q}[e_1; e_0; e_1] \implies [e_1; e_0; e_1] = \lambda_{2,3}[e_0; e_0; e_2]$.

◦ Case $N = 3$.

Plugging $i_1 = 1, i_2 = 2, i_3 = 0$ into (u6'), we get

$$[e_1; e_3; e_{-1}] + [e_2; e_2; e_{-1}] + [e_2; e_1; e_0] + [e_0; e_3; e_0] + [e_0; e_2; e_1] = 0.$$

Applying (12), we get: $-(q+2+q^{-1})[e_2; e_0; e_1] - (q+q^{-1})[e_1; e_0; e_2] - (1+\frac{q^2-q}{q^3-1})[e_0; e_0; e_3] = 0$. On the other hand, applying $\text{ad}(\mathbf{h}_1)$ to $(q+1)^2[e_1; e_0; e_1] + q[e_0; e_0; e_2] = 0$ (case $N = 2$), we get $(q+1)^2[e_2; e_0; e_1] + (q^2+3q+1)[e_1; e_0; e_2] + (q-\frac{q^2-q^3}{q^3-1})[e_0; e_0; e_3] = 0$. These two linear combinations of $[e_2; e_0; e_1], [e_1; e_0; e_2], [e_0; e_0; e_3]$ are not proportional for $q \neq \sqrt{1}$. Therefore, we can eliminate $[e_2; e_0; e_1]$, which proves that $[e_1; e_0; e_2]$ is a multiple of $[e_0; e_0; e_3]$.

◦ Case $N = k+2, k > 1$.

By an induction assumption $[e_1; e_0; e_k] - \lambda_{k+1,3}[e_0; e_0; e_{k+1}] = 0$. Applying $\text{ad}(\mathbf{h}_1)$, we get $([e_2; e_0; e_k] + [e_1; e_1; e_k] + [e_1; e_0; e_{k+1}]) - \lambda_{k+1,3}([e_1; e_0; e_{k+1}] + [e_0; e_1; e_{k+1}] + [e_0; e_0; e_{k+2}]) = 0$. Note also that

$$(\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2))([e_{i_1}; e_{i_2}; e_{i_3}]) = [e_{i_1+1}; e_{i_2+1}; e_{i_3}] + [e_{i_1+1}; e_{i_2}; e_{i_3+1}] + [e_{i_1}; e_{i_2+1}; e_{i_3+1}].$$

By an induction assumption $[e_1; e_0; e_{k-1}] = \lambda_{k,3}[e_0; e_0; e_k]$. Applying $\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2)$, we get

$$([e_2; e_1; e_{k-1}] + [e_2; e_0; e_k] + [e_1; e_1; e_k]) - \lambda_{k,3}([e_1; e_1; e_k] + [e_1; e_0; e_{k+1}] + [e_0; e_1; e_{k+1}]) = 0.$$

Applying (12), we get two linear combinations of $[e_2; e_0; e_k], [e_1; e_0; e_{k+1}], [e_0; e_0; e_{k+2}]$ being zero. It is a routine verification to check that they are not proportional for $q \neq \sqrt{1}$. Therefore, we can eliminate $[e_2; e_0; e_k]$, which proves that $[e_1; e_0; e_{k+1}]$ is a multiple of $[e_0; e_0; e_{k+2}]$.

• Case $j = n > 3$.

Analogously to the previous case, it suffices to show that $[e_1; e_0; \dots; e_0; e_{N-1}]_n$ is a multiple of $[e_0; \dots; e_0; e_N]_n$. This is equivalent to

$$[e_1; \dots; e_0; e_{N-1}]_n = \lambda_{N,n} \cdot [e_0; \dots; e_0; e_N]_n, \quad \lambda_{N,n} = \frac{(q^{N-1}-1)^{n-2}(q^{N-1}-q^{n-1})}{(q^N-1)^{n-1}},$$

the constant being computed by comparing the images under θ_m .

We will need the following *multiple* counterpart of (u6) (follows from Proposition 7.5):

$$(u7n) \quad [e_0; e_1; e_0; \dots; e_0; e_{-1}]_n = 0.$$

This equality together with the relation (u4) implies

$$(u7'n) \quad \text{Sym}[e_{i_1}; e_{i_2+1}; e_{i_3}; \dots; e_{i_{n-1}}; e_{i_n-1}]_n = 0 \quad \forall i_1, \dots, i_n \in \mathbb{Z}.$$

Now we proceed to the proof of the aforementioned result by an induction on N .

◦ *Case $N = 2$.*

Applying $\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2)$ to (u7n), we get

$$[e_1, \text{ad}(\mathbf{h}_1)[e_1; e_0; \dots; e_{-1}]_{n-1}] + [e_0; (\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2))[e_1; e_0; \dots; e_{-1}]_{n-1}] = 0.$$

By the induction assumption for n , this has a form $a_n \cdot [e_1; e_0; \dots; e_1]_n + b_n \cdot [e_0; \dots; e_0; e_2]_n = 0$.

Computing the images under θ_m , we find $a_n = \frac{(-1)^{n-3}(q^n-1)^2}{q^{n-1}(q-1)}$, which is non-zero for $q \neq \sqrt{1}$.

◦ *Case $N = 3$.*

Applying $\text{ad}(\mathbf{h}_1)$ to $[e_1; \dots; e_0; e_1]_n - \lambda_{2,n}[e_0; \dots; e_0; e_2]_n = 0$, we get

$$[e_2; e_0; \dots; e_1]_n + [e_1, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_1]_{n-1}] - \lambda_{2,n}[e_1; \dots; e_0; e_2]_n - \lambda_{2,n}[e_0, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_2]_{n-1}] = 0.$$

Applying the induction step for $j = n - 1$, this equation can be simplified to

$$[e_2; e_0; \dots; e_0; e_1]_n + c_n \cdot [e_1; e_0; \dots; e_0; e_2]_n + d_n \cdot [e_0; e_0; \dots; e_0; e_3]_n = 0.$$

Computing the images under θ_m , one gets $c_n = \frac{(q-1)^{n-2}}{(q^2-1)^{n-1}}(q^n + 2q^{n-1} - 2q - 1)$.

On the other hand, applying $\text{ad}(\mathbf{h}_1)\text{ad}(\mathbf{h}_2) - \text{ad}(\mathbf{h}_3)$ to (u7n), we get

$$[e_2; \text{ad}(\mathbf{h}_1)[e_1; \dots; e_{-1}]_{n-1}] + [e_1, \text{ad}(\mathbf{h}_2)[e_1, \dots, e_{-1}]_{n-1}] + [e_0; (\text{ad}(\mathbf{h}_1)\text{ad}(\mathbf{h}_2) - \text{ad}(\mathbf{h}_3))[e_1, \dots, e_{-1}]_{n-1}] = 0.$$

Applying the induction step for $j = n - 1$, this equation can be simplified to

$$a'_n \cdot [e_2; e_0; \dots; e_0; e_1]_n + c'_n \cdot [e_1; e_0; \dots; e_0; e_2]_n + d'_n \cdot [e_0; e_0; \dots; e_0; e_3]_n = 0.$$

By computing the images under θ_m , one gets the following formulas

$$a'_n = \frac{(q^{n-1} - 1)^2}{(q - 1)^2}, c'_n = \frac{(-1)^n(q - 1)^{n-4}(q^{n-1} - 1)^2(q^{n-1} + 1)}{q^{n-2}(q + 1)(q^2 - 1)^{n-2}}.$$

It remains to notice that $c'_n \neq a'_n c_n$ for $q \neq \sqrt{1}$. Therefore, eliminating $[e_2; e_0; \dots; e_0; e_1]_n$, we see that $[e_1; e_0; \dots; e_0; e_2]_n$ is a multiple of $[e_0; e_0; \dots; e_0; e_3]_n$.

◦ *Case $N = k + 2, k > 1$.*

By the induction: $[e_1; e_0; \dots; e_k]_n - \lambda_{k+1,n}[e_0; \dots; e_0; e_{k+1}]_n = 0$. Applying $\text{ad}(\mathbf{h}_1)$, we get

$$[e_2; e_0; \dots; e_k]_n + [e_1, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_k]_{n-1}] - \lambda_{k+1,n}([e_1; e_0; \dots; e_{k+1}]_n + [e_0, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_{k+1}]_{n-1}]) = 0.$$

By an induction assumption on length $j < n$ commutators, this equality can be simplified to

$$[e_2; e_0; \dots; e_k]_n + v_n \cdot [e_1; e_0; \dots; e_{k+1}]_n + w_n \cdot [e_0; e_0; \dots; e_{k+2}]_n = 0.$$

By computing the images under θ_m , we find $v_n = \frac{(q^k-1)^{n-2}(q^{n+k}-2q^{k+1}-2q^{n-1}+q^n+q^k+1)}{(q^{k+1}-1)^{n-1}(q-1)}$.

On the other hand, by an induction assumption: $[e_1; \dots; e_0; e_{k-1}]_n - \lambda_{k,n}[e_0; \dots; e_0; e_k]_n = 0$.

Applying $\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2)$, we get

$$[e_2, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_{k-1}]_{n-1}] + [e_1, (\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2))[e_0; \dots; e_{k-1}]_{n-1}] - \lambda_{k,n}([e_1, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_k]_{n-1}] + [e_0, (\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2))[e_0; \dots; e_k]_{n-1}]) = 0.$$

By an induction assumption on length $j < n$ commutators, this equality can be simplified to

$$u'_n \cdot [e_2; e_0; \dots; e_k]_n + v'_n \cdot [e_1; e_0; \dots; e_{k+1}]_n + w'_n \cdot [e_0; e_0; \dots; e_{k+2}]_n = 0.$$

By computing the images under θ_m , we find

$$u'_n = \frac{(q^{n-1} - 1)(q^{k-1} - 1)^{n-2}}{(q - 1)(q^k - 1)^{n-2}}, v'_n = \frac{(q^{k-1} - 1)^{n-2}(q^{n-1} - 1)}{(q^{k+1} - 1)^{n-2}(q - 1)} \left(\frac{q^{n-1} - q}{q^2 - 1} - \frac{q^{k-1} - q^{n-1}}{q^k - 1} \right).$$

It remains to notice that $v'_n \neq u'_n v_n$ for $q \neq \sqrt{1}$. Therefore, eliminating $[e_2; e_0; \dots; e_0; e_k]_n$, we see that $[e_1; e_0; \dots; e_0; e_{k+1}]_n$ is a multiple of $[e_0; e_0; \dots; e_0; e_{k+2}]_n$.

This completes the proof of $\dim(\ddot{u}_{h_0})_{i,j} \leq 1$ for $j > 0$. The case $j < 0$ is analogous.

C.2. Proof of Theorem 5.9. We prove that θ_a is an isomorphism of \mathbb{C} -algebras for $h_0 \neq 0$.

As mentioned in Section 5.5, all the defining relations of the algebra $\check{Y}'_{h_0}(\mathfrak{gl}_1)$ are of Lie-type. Therefore, it is the universal enveloping algebra of the Lie algebra \check{y}'_{h_0} generated by $\{e_j, f_j, \psi_j\}$ with the same defining relations. Moreover, \check{y}'_{h_0} is a 1-dimensional central extension of the Lie-algebra \check{y}_{h_0} generated by $\{e_j, f_j, \psi_{j+1}\}_{j \in \mathbb{Z}_+}$ with the following defining relations:

$$\begin{aligned}
(y0) \quad & [\psi_k, \psi_l] = 0, \\
(y1) \quad & [e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] - h_0^2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = 0, \\
(y2) \quad & [f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] - h_0^2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = 0, \\
(y3) \quad & [e_0, f_0] = 0, [e_i, f_j] = \psi_{i+j} \text{ for } i+j > 0, \\
(y4) \quad & [\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}] - h_0^2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) = 0, \\
(y4') \quad & [\psi_1, e_j] = 0, [\psi_2, e_j] = 2h_0^2 e_j, \\
(y5) \quad & [\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}] - h_0^2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) = 0, \\
(y5') \quad & [\psi_1, f_j] = 0, [\psi_2, f_j] = -2h_0^2 f_j, \\
(y6) \quad & \text{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0, \text{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0.
\end{aligned}$$

Hence, it suffices to check that the corresponding homomorphism $\theta_a : \check{y}_{h_0} \rightarrow \mathfrak{D}_{h_0}$ defined by

$$\theta_a : e_j \mapsto x^j \partial, f_j \mapsto -\partial^{-1} x^j, \psi_{j+1} \mapsto ((x - h_0)^{j+1} - x^{j+1}) \partial^0,$$

is an isomorphism of the \mathbb{C} -Lie algebras for $h_0 \neq 0$. The surjectivity of θ_a is clear.

The Lie algebra \check{y}_{h_0} is \mathbb{Z} -graded, where we set $\deg_2(e_j) = 1, \deg_2(f_j) = -1, \deg_2(\psi_{j+1}) = 0$. It is also \mathbb{Z}_+ -filtered with $\deg_1(e_j) = j, \deg_1(f_j) = j, \deg_1(\psi_{j+1}) = j$. The Lie algebra \mathfrak{D}_{h_0} is also \mathbb{Z} -graded with $\deg_2(x^i \partial^j) = j$ and \mathbb{Z}_+ -filtered with $\deg_1(x^i \partial^j) = i$. Moreover, θ_a intertwines those gradings and filtrations. Note that $\dim(\mathfrak{D}_{h_0})_{\leq i, j} = \dim(\mathfrak{D}_{h_0})_{\leq i-1, j} + 1$. Let $\check{y}_{h_0}^{\geq 0}$ be the Lie algebra generated by $\{e_j, \psi_k\}$ with the defining relations (y0, y1, y4, y4', y6). The result follows from the following inequality: $\dim(\check{y}_{h_0}^{\geq 0})_{\leq i, j} - \dim(\check{y}_{h_0}^{\geq 0})_{\leq i-1, j} \leq 1$.

Consider a subspace $V_{\leq i, n} \subset (\check{y}_{h_0}^{\geq 0})_{\leq i, n}$ spanned by $\{[e_{i_1}; \dots; e_{i_n}] | i_1 + \dots + i_n \leq i + n - 1\}$. Note that $[e_0; \dots; e_0; e_{n+i-1}]_n \in (\check{y}_{h_0}^{\geq 0})_{\leq i, n} \setminus (\check{y}_{h_0}^{\geq 0})_{\leq i-1, n}$. The above inequality follows from

$$(\diamond_{i, j}) \quad \dim V_{\leq i, j} - \dim V_{\leq i-1, j} \leq 1.$$

The relations (y4, y4') imply the following result: $[\psi_k, e_j] - k(k-1)h_0^2 e_{j+k-2} \in V_{< j+k-2, 1}$.

• *Case $j = 1, 2$.*

The subspace $V_{\leq i, 1}$ is spanned by $\{e_0, e_1, \dots, e_i\}$. The inequality $(\diamond_{i, 1})$ follows.

The subspace $V_{\leq N, 2}$ is spanned by $\{[e_{i_1}, e_{i_2}] | i_1 + i_2 \leq N + 1\}$. The relation (y1) implies:

$$[e_{i+2+k}, e_{i+1-k}] - (2k+1)[e_{i+2}, e_{i+1}] \in V_{\leq 2i+1, 2}, [e_{i+2+k}, e_{i-k}] - (k+1)[e_{i+2}, e_i] \in V_{\leq 2i, 2}.$$

These formulas can be unified in the following way:

$$(13) \quad [e_i, e_j] - \frac{j-i}{i+j} [e_0, e_{i+j}] \in V_{\leq i+j-2, 2}.$$

Hence, $V_{\leq i, 2}/V_{\leq i-1, 2}$ is spanned by the image of $[e_0, e_{i+1}]$. The inequality $(\diamond_{i, 2})$ follows.

• *Case $j = 3$.*

Our goal is to show that $[e_{i_1}; e_{i_2}; e_{i_3}]$ is a multiple of $[e_0; e_0; e_{i_1+i_2+i_3}]$ modulo $V_{\leq i_1+i_2+i_3-3, 3}$, which will be denoted by $[e_{i_1}; e_{i_2}; e_{i_3}] \sim [e_0; e_0; e_{i_1+i_2+i_3}]$. By $(\diamond_{\bullet, 2})$, we can assume $i_2 = 0$.

To proceed, we introduce the elements $\mathbf{h}_1, \mathbf{h}_2 \in \ddot{y}_{h_0}^{\geq 0}$ by $\mathbf{h}_1 := \frac{\psi_3}{6h_0^2}, \mathbf{h}_2 := \frac{\psi_4 + h_0^2\psi_2}{12h_0^2}$. According to (y4, y4'), we have $[\mathbf{h}_1, e_j] = e_{j+1}, [\mathbf{h}_2, \psi_j] = e_{j+2}$. Same reasoning as in Appendix B.1 shows that applying $\text{ad}(\mathbf{h}_1)$ to $[e_k; e_0; e_l] \sim [e_0; e_0; e_{k+l}]$ implies $[e_{k+1}; e_0; e_l] \sim [e_0; e_0; e_{k+l+1}]$. Therefore, it remains to prove $[e_1; e_0; e_{N-1}] \sim [e_1; e_0; e_N]$. Computing the images of both commutators under θ_a , we see that $[e_1; e_0; e_{N-1}] \equiv \beta_{N,3}[e_0; e_0; e_N]$ for $\beta_{N,3} = \frac{N-4}{N}$. We write $a \equiv b$ if $a - b \in V_{\leq i-1, j}$ for $a, b \in V_{\leq i, j}$.

◦ *Case $N = 1, 2$.*

We have $[e_1; e_0; e_0] = 0 = [e_0; e_0; e_1]$. Applying $\text{ad}(\mathbf{h}_1)$ to this, we get $[e_1; e_0; e_1] = -[e_0; e_0; e_2]$.

◦ *Cases $N = k + 1, k > 1$.*

By an induction assumption: $[e_1; e_0; e_{k-1}] \equiv \beta_{k,3}[e_0; e_0; e_k]$. Applying $\text{ad}(\mathbf{h}_1)$, we get

$$[e_2; e_0; e_{k-1}] + [e_1; e_1; e_{k-1}] + [e_1; e_0; e_k] \equiv \beta_{k,3}([e_1; e_0; e_k] + [e_0; e_1; e_k] + [e_0; e_0; e_{k+1}]).$$

Applying (13), we get $[e_2; e_0; e_{k-1}] + \frac{k+2}{k}[e_1; e_0; e_k] \sim [e_0; e_0; e_{k+1}]$. On the other hand, we have $[e_1; e_0; e_{k-2}] \equiv \beta_{k-1,3}[e_0; e_0; e_{k-1}]$. Applying $\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2)$, we get

$$[e_2; e_1; e_{k-2}] + [e_2; e_0; e_{k-1}] + [e_1; e_1; e_{k-1}] \equiv \beta_{k-1,3}([e_1; e_1; e_{k-1}] + [e_1; e_0; e_k] + [e_0; e_1; e_k]).$$

Applying (13), we get $\frac{2(k-2)}{k-1}[e_2; e_0; e_{k-1}] + \frac{8-k}{k}[e_1; e_0; e_k] \sim [e_0; e_0; e_{k+1}]$. Comparing those two linear combinations of $[e_2; e_0; e_{k-1}], [e_1; e_0; e_k]$, we get $[e_1; e_0; e_k] \sim [e_0; e_0; e_{k+1}]$, unless $k = 3$. We will consider this particular case in the greater generality below.

• *Case $j > 3$.*

Analogously to the $j = 3$ case it suffices to show that $[e_1; e_0; \dots; e_{N-1}]_n \sim [e_0; \dots; e_0; e_N]_n$. This is equivalent to $[e_1; \dots; e_0; e_{N-1}]_n \equiv \beta_{N,n}[e_0; \dots; e_0; e_N]_n$, $\beta_{N,n} = \frac{N-2n+2}{N}$, the constant being computed by comparing the images under θ_a .

We will need the following *multiple* counterpart of (y6) (follows from Proposition 7.8):

$$(y7n) \quad [e_0; \dots; e_0; e_{n-2}]_n = 0.$$

Now we proceed to the proof of the aforementioned result by an induction on N .

◦ *Case $N \leq n - 1$.*

If $N < n - 1$, then $[e_0; \dots; e_0; e_{N-1}]_{n-1} = 0 = [e_0; \dots; e_0; e_N]_n$.

Applying $\text{ad}(\mathbf{h}_1)$ to $[e_0; \dots; e_0; e_{n-2}]_n = 0$, we get $[e_1; \dots; e_0; e_{n-2}]_n \sim [e_0; \dots; e_0; e_{n-1}]_n$.

◦ *Case $N = k + 1, k > n - 2$.*

Applying $\text{ad}(\mathbf{h}_1)$ to $[e_1; \dots; e_0; e_{k-1}]_n \equiv \beta_{k,n}[e_0; \dots; e_0; e_k]_n$, we get

$$[e_2; \dots; e_0; e_{k-1}] + [e_1, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_{k-1}]_{n-1}] \equiv \beta_{k,n}([e_1; \dots; e_0; e_k]_n + [e_0, \text{ad}(\mathbf{h}_1)[e_0; \dots; e_0; e_k]_{n-1}]).$$

By an induction assumption for $j = n - 1$, this is equivalent to

$$[e_2; \dots; e_0; e_{k-1}]_n + \frac{(n-2)k - (n-1)(n-4)}{k}[e_1; \dots; e_0; e_k]_n \equiv [e_0; \dots; e_0; e_{k+1}]_n.$$

Applying $\text{ad}(\mathbf{h}_1)^2 - \text{ad}(\mathbf{h}_2)$ to $[e_1; \dots; e_0; e_{k-2}]_n \equiv \beta_{k-1,n}[e_0; \dots; e_0; e_{k-1}]_n$ together with an induction assumption for $j = n - 1$, we get

$$P[e_2; \dots; e_0; e_{k-1}]_n + Q[e_1; \dots; e_0; e_k]_n \equiv [e_0; \dots; e_0; e_{k+1}]_n,$$

with $P = \frac{(n-1)(k-n+1)}{k-1}, Q = \frac{n-1}{2k(k-1)}(k^2(n-4) - k(2n^2 - 13n + 12) + (n^3 - 9n^2 + 18n - 8))$.

Comparing those two linear combinations, we get $[e_1; \dots; e_0; e_k]_n \sim [e_0; \dots; e_0; e_{k+1}]_n$ for $k \neq n$.

It remains to consider the case $k = n$. Choose $\mathbf{h}_3 \equiv \frac{\psi_5}{20h_0^2}$ such that $[\mathbf{h}_3, e_j] = e_{j+3}$. Applying $\text{ad}(\mathbf{h}_1)\text{ad}(\mathbf{h}_2) - \text{ad}(\mathbf{h}_3)$ to $[e_0; \dots; e_0; e_{n-2}]_n = 0$, we get

$$[e_2; \dots; e_0; e_{n-1}]_n + \frac{2}{n}[e_1; \dots; e_0; e_n]_n \sim [e_0; \dots; e_0; e_{n+1}]_n.$$

This equivalence together with the previous one implies $[e_1; \dots; e_0; e_n]_n \sim [e_0; \dots; e_0; e_{n+1}]_n$.

APPENDIX D. PROOF OF PROPOSITION 7.5

Proposition 7.5 follows from Theorem 7.4(c) and the following two lemmas.

Lemma D.1. *The elements L_j^m belong to \mathcal{A}^m .*

Lemma D.2. *The elements $\{L_j^m\}_{j \geq 1}$ are algebraically independent.*

Proof of Lemma D.1.

According to Theorem 7.4(a), it suffices to show that $\partial^{(\infty, k)} L_n^m$ exist for all k . Note that:

$$(14) \quad L_n^m = \text{Sym}_{\mathfrak{S}_n} \left\{ \left(\sum_{l=0}^{n-2} (-1)^l \binom{n-2}{l} \frac{x_1}{x_{n-l}} - \sum_{l=0}^{n-2} (-1)^l \binom{n-2}{l} \frac{x_n}{x_{n-1-l}} \right) \prod_{i < j} \omega^m(x_i, x_j) \right\}.$$

Our goal is to show that the RHS of (14) has a finite limit as $x_{n-k+1} \mapsto \xi \cdot x_{n-k+1}, \dots, x_n \mapsto \xi \cdot x_n$ with $\xi \rightarrow \infty$. Note that $\frac{x_{\sigma(i)}}{x_{\sigma(j)}}$ has a finite limit as $\xi \rightarrow \infty$, unless $\sigma(j) \leq n-k < \sigma(i)$, in which case it has a linear growth. On the other hand, $\omega^m(x_i, x_j)$ has a finite limit as $\xi \rightarrow \infty$. Moreover: $\omega^m(\xi \cdot x, y) = 1 + O(\xi^{-1})$, $\omega^m(y, \xi \cdot x) = 1 + O(\xi^{-1})$ as $\xi \rightarrow \infty$. This reduces to proving $A_1 - A_2 = 0$, where A_1, A_2 are given by

$$A_1 = \frac{1}{n!} \sum_{s=n-k+1}^n \sum_{\sigma \in \mathfrak{S}_n}^{\sigma(1)=s} \sum_{l: \sigma(n-l) \leq n-k} (-1)^l \binom{n-2}{l} \frac{x_s}{x_{\sigma(n-l)}} \prod_{i < j}^{j \leq n-k} \omega_{\sigma}^m(x_i, x_j) \prod_{i < j}^{n-k < i} \omega_{\sigma}^m(x_i, x_j),$$

$$A_2 = \frac{1}{n!} \sum_{s=n-k+1}^n \sum_{\sigma \in \mathfrak{S}_n}^{\sigma(n)=s} \sum_{l: \sigma(n-l-1) \leq n-k} (-1)^l \binom{n-2}{l} \frac{x_s}{x_{\sigma(n-l-1)}} \prod_{i < j}^{j \leq n-k} \omega_{\sigma}^m(x_i, x_j) \prod_{i < j}^{n-k < i} \omega_{\sigma}^m(x_i, x_j).$$

Here we set $\omega_{\sigma}^m(x_i, x_j) = \omega^m(x_i, x_j)$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $\omega_{\sigma}^m(x_i, x_j) = \omega^m(x_j, x_i)$ otherwise.

If $k = 1$, then $s = n$ in both sums and the map $(\sigma, l) \mapsto (\sigma', l)$ with $\sigma'(i) := \sigma(i+1)$ establishes a bijection between the equal summands in A_1 and A_2 , so that $A_1 - A_2 = 0$.

For $k > 1$, there is no such bijection. Instead, we show $A_1 = 0$ (equality $A_2 = 0$ is analogous). Let us group the summands in A_1 according to s , $\sigma(n-l)$ and also the ordering of $\{x_1, \dots, x_{n-k}\}$ and $\{x_{n-k+1}, \dots, x_n\}$, which are given by elements $\sigma_1 \in \mathfrak{S}_{n-k}$ and $\sigma_2 \in \mathfrak{S}_k$. Define

$$\omega_{\sigma_1, \sigma_2}^m(x_1; \dots; x_n) := \prod_{i < j}^{j \leq n-k} \omega_{\sigma_1}^m(x_i, x_j) \cdot \prod_{i < j}^{n-k < i} \omega_{\sigma_2}^m(x_i, x_j).$$

Then A_1 can be written in the form

$$A_1 = \frac{1}{n!} \sum_{t \leq n-k} \sum_{\sigma_1 \in \mathfrak{S}_{n-k}} \sum_{\sigma_2 \in \mathfrak{S}_k} A_{t, \sigma_1, \sigma_2} \frac{x_{\sigma_2(1)}}{x_t} \omega_{\sigma_1, \sigma_2}^m(x_1; \dots; x_n), \quad A_{t, \sigma_1, \sigma_2} \in \mathbb{Z}.$$

We claim that all those constants $A_{t, \sigma_1, \sigma_2}$ are zero. As an example, we compute $A_{t, 1_{n-k}, 1_k}$:

$$A_{t, 1_{n-k}, 1_k} = \sum_{l=n-k-t}^{n-t-1} (-1)^l \binom{n-2}{l} \binom{l}{n-k-t} \binom{n-l-2}{t-1} =$$

$$\frac{(-1)^{n-k-t} (n-2)!}{(t-1)! (k-1)! (n-k-t)!} (1-1)^{k-1} = 0 \text{ as } k > 1.$$

Analogously $A_{t, \sigma_1, \sigma_2} = 0$ for any t, σ_1, σ_2 . Hence, $A_1 = 0$ and the result follows. \square

Proof of Lemma D.2.

The elements L_j^m correspond to nonzero multiples of $\theta_{j,0}$ via $S^m \simeq \tilde{\mathcal{E}}^+$. An algebraic independence of $\{\theta_{j,0}\}_{j>0}$ follows from an analogue of Proposition A.1(b) applied to $\ddot{\mathbf{U}}_{q_1, q_2, q_3}$. \square

APPENDIX E. PROOF OF THEOREMS 8.5, 8.6

E.1. Sketch of the proof of Theorem 8.5.

According to the fixed point formula, we have $v_r^K = \sum_{\bar{\lambda}} a_{\bar{\lambda}} \cdot [\bar{\lambda}]$, $a_{\bar{\lambda}} = \prod_{w \in T_{\bar{\lambda}} M(r, |\bar{\lambda}|)} (1-w)^{-1}$. Hence, we need to show that for any r -partition $\bar{\lambda}$ the following equality holds:

$$(15) \quad C_{j, -n} = \sum_{\bar{\lambda}'} \frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}} \cdot K_{-n}^{(m; j)}_{|[\bar{\lambda}', \bar{\lambda}]},$$

where the sum is over all r -partitions $\bar{\lambda}'$ such that $\bar{\lambda} \subset \bar{\lambda}'$ and $|\bar{\lambda}'| = |\bar{\lambda}| + n$.

For such a pair of r -partitions $(\bar{\lambda}, \bar{\lambda}')$, define a collection of positive integers

$$(16) \quad j_{1,1} \leq j_{1,2} \leq \dots \leq j_{1,l_1}, \quad j_{2,1} \leq j_{2,2} \leq \dots \leq j_{2,l_2}, \quad \dots, \quad j_{r,1} \leq j_{r,2} \leq \dots \leq j_{r,l_r}, \quad \sum_{i=1}^r l_i = n$$

via the following equality:

$$\bar{\lambda}' = \bar{\lambda} + \square_{j_{1,1}}^1 + \dots + \square_{j_{1,l_1}}^1 + \square_{j_{2,1}}^2 + \dots + \square_{j_{2,l_2}}^2 + \dots + \square_{j_{r,1}}^r + \dots + \square_{j_{r,l_r}}^r.$$

We also introduce the sequence of r -partitions $\bar{\lambda} = \bar{\lambda}^{[0]} \subset \bar{\lambda}^{[1]} \subset \dots \subset \bar{\lambda}^{[n]} = \bar{\lambda}'$, where $\bar{\lambda}^{[r]}$ is obtained from $\bar{\lambda}$ by adding the first q boxes from above. For $1 \leq q \leq n$, the q -th box from above has a form $\square_{j_{sq}, i_q}^{s_q}$. We denote its character by $\chi(q)$.

For any $F \in (S_n^m)^{\text{opp}}$, we get the following formula for the matrix coefficient $F_{|[\bar{\lambda}', \bar{\lambda}]}$:

$$F_{|[\bar{\lambda}', \bar{\lambda}]} = \frac{F(\chi(1), \dots, \chi(n))}{\prod_{a < b} \omega^m(\chi(a), \chi(b))} \cdot \prod_{q=1}^n f_{0|[\bar{\lambda}^{[q]}, \bar{\lambda}^{[q-1]}]}.$$

In particular, we have

$$K_{-n}^{(m; j)}_{|[\bar{\lambda}', \bar{\lambda}]} = \prod_{1 \leq a < b \leq n} \frac{(\chi(a) - \chi(b))(\chi(b) - t_1 \chi(a))}{(\chi(a) - t_2 \chi(b))(\chi(a) - t_3 \chi(b))} \cdot \prod_{q=1}^n \chi(q)^j \cdot \prod_{q=1}^n f_{0|[\bar{\lambda}^{[q]}, \bar{\lambda}^{[q-1]}]}.$$

As an immediate consequence of this formula, we get $K_{-n}^{(m; j)}_{|[\bar{\lambda}', \bar{\lambda}]} = 0$ if $\bar{\lambda}' \setminus \bar{\lambda}$ contains two boxes in the same row of its i -th component, $1 \leq i \leq r$. Therefore, the sum in (15) should be taken only over those collections $\{j_{1,1}, \dots, j_{r,l_r}\}$ from (16) which satisfy strict inequalities.

We also split $\frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}}$ into the product over consequent pairs: $\frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}} = \prod_{q=1}^n \frac{a_{\bar{\lambda}^{[q]}}}{a_{\bar{\lambda}^{[q-1]}}}$. According to the Bott-Lefschetz fixed point formula, we have

$$\frac{a_{\bar{\lambda}^{[q]}}}{a_{\bar{\lambda}^{[q-1]}}} \cdot f_{0|[\bar{\lambda}^{[q]}, \bar{\lambda}^{[q-1]}]} = e_{-r|[\bar{\lambda}^{[q-1]}, \bar{\lambda}^{[q]}]}.$$

For a pair of two r -partitions $(\bar{\mu}, \bar{\mu}')$ such that $\bar{\mu}' = \bar{\mu} + \square_j^l$, the matrix coefficient $e_{-r|[\bar{\mu}, \bar{\mu}']}$ was computed in Lemma 3.3(a):

$$e_{-r|[\bar{\mu}, \bar{\mu}']} = -\frac{t_1^{1-r} t_2}{1 - t_1} \prod_{a=1}^r \frac{1}{\chi_j^{(l)} - t_1^{-1} t_2^{L_a} \chi_a^{-1}} \prod_{(a,k) \neq (l,j)}^{k \leq L_a} \frac{\chi_j^{(l)} - t_2 \chi_k^{(a)}}{\chi_j^{(l)} - \chi_k^{(a)}},$$

where L_a is chosen to satisfy $L_a \geq \mu_1^{a*} + 1$.

Combining these formulas together, we finally get

$$\frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}} \cdot K_{-n}^{(m; j)}_{|[\bar{\lambda}', \bar{\lambda}]} = \prod_{1 \leq q \leq n} \left\{ \frac{(-t_1 t_2)^q}{1 - t_1} \cdot \prod_{a=1}^r \frac{1}{\chi(q) - t_2^{L_a} \chi_a^{-1}} \cdot \prod \frac{\chi(q) - t_1 t_2 \chi_k^{(a)}}{\chi(q) - t_1 \chi_k^{(a)}} \cdot \chi(q)^j \right\},$$

the second product is over pairs $(a, k) \notin \{(1, j_{1,1}), \dots, (r, j_{r,l_r})\}$, $k \leq L_a$ with $L_a \geq \lambda_1^{a*} + n$.

Let us denote the RHS of this equality by $C_{\mathbf{j}}$, where $\mathbf{j} = \{j_{1,1}, \dots, j_{r,l_r}\}$ is determined by (16). Note that $C_{\mathbf{j}} = 0$ if the corresponding r -partition $\bar{\lambda}'$ fails to be a collection of r Young diagrams. Hence, (15) reduces to $C_{j,-n} = \sum C_{\mathbf{j}}$, the sum over all \mathbf{j} from (16) with strict inequalities.

It is easy to check that the sum $\sum C_{\mathbf{j}}$ has no poles for $j \geq 0$. Together with the degree computation, we see that it is independent of $\bar{\lambda}$ for $0 \leq j \leq r$. Thus v_r^K is indeed an eigenvector with respect to $K_{-n}^{(m;j)}$. To compute its eigenvalue, we evaluate $\sum C_{\mathbf{j}}$ at $\bar{\lambda} = \bar{\emptyset}$. This sum is actually over all partitions (l_1, \dots, l_r) of n with $j_{a,b} = b$. The total sum equals

$$\begin{aligned} & \frac{(-t_1 t_2)^{n(n+1)/2}}{(1-t_1)^n} \sum_{l_1+\dots+l_r=n} \prod_{a,b=1}^r \frac{1}{(\chi_b^{-1} - t_2^{l_a} \chi_a^{-1}) \cdots (t_2^{l_b-1} \chi_b^{-1} - t_2^{l_a} \chi_a^{-1})} \prod_{b=1}^r (t_2^{l_b(l_b-1)/2} \chi_b^{-l_b})^j = \\ & \frac{(-t_1 t_2)^{n(n+1)/2} (\chi_1 \cdots \chi_r)^n}{(1-t_1)^n} \sum_{l_1+\dots+l_r=n} \prod_{a,b=1}^r \frac{1}{(\chi_a - t_2^{l_a} \chi_b) \cdots (\chi_a - t_2^{l_a-l_b+1} \chi_b)} \prod_{b=1}^r (t_2^{-l_b(l_b-1)/2} \chi_b^{l_b})^{r-j}. \end{aligned}$$

It is straightforward to check that for $0 \leq j \leq r-1$, the sum from the above equality is a rational function in χ_a with no poles. Together with the degree estimate, we see that it is independent of $\{\chi_a\}$. To compute this constant we let $\chi_1 \rightarrow \infty$. Then the only nonzero contribution comes from the collection $(l_1, l_2, \dots, l_r) = (n, 0, \dots, 0)$ and the result equals $C_{j,-n}$.

For $j = r$, the whole expression above has no poles and is of total degree ≤ 0 ; therefore, it is independent of χ_a . To compute this constant we let $\chi_1 \rightarrow \infty$. The only nonzero contributions come from those (l_1, \dots, l_r) with $l_1 = 0$. For those we let $\chi_2 \rightarrow \infty$, etc. The result follows by straightforward computations.

E.2. Sketch of the proof of Theorem 8.6.

According to the fixed point formula, we have $v_r^H = \sum_{\bar{\lambda}} b_{\bar{\lambda}} \cdot [\bar{\lambda}]$, $b_{\bar{\lambda}} = \prod_{w \in T_{\bar{\lambda}} M(r, |\bar{\lambda}|)} w^{-1}$. Hence, we need to show that for any r -partition $\bar{\lambda}$ the following equality holds:

$$(17) \quad D_{j,-n} = \sum_{\bar{\lambda}'} \frac{b_{\bar{\lambda}'}}{b_{\bar{\lambda}}} \cdot K_{-n}^{(a;j)}|_{[\bar{\lambda}', \bar{\lambda}]},$$

where the sum is over all r -partitions $\bar{\lambda}'$ such that $\bar{\lambda} \subset \bar{\lambda}'$ and $|\bar{\lambda}'| = |\bar{\lambda}| + n$.

Analogously to the K-theoretical case, we have:

$$\frac{b_{\bar{\lambda}'}}{b_{\bar{\lambda}}} \cdot K_{-n}^{(a;j)}|_{[\bar{\lambda}', \bar{\lambda}]} = \prod_{1 \leq q \leq n} \left\{ \frac{(-1)^{q+r}}{s_1} \cdot \prod_{a=1}^r \frac{1}{\chi(q) - L_a s_2 + x_a} \cdot \prod \frac{\chi(q) - x_k^{(a)} - s_1 - s_2}{\chi(q) - x_k^{(a)} - s_1} \cdot \chi(q)^j \right\},$$

the second product is over pairs $(a, k) \notin \{(1, j_{1,1}), \dots, (r, j_{r,l_r})\}$, $k \leq L_a$ with $L_a \geq \lambda_1^{a*} + n$.

Let us denote the RHS of this equality by $D_{\mathbf{j}}$. Then $\sum_{\mathbf{j}} D_{\mathbf{j}}$ is a rational function in $x_k^{(a)}$ with no poles for $j \geq 0$. The degree estimate implies that for $j \leq r$ it is independent of $x_k^{(a)}$. Thus v_r^H is indeed an eigenvector with respect to $\{K_{-n}^{(a;j)}\}_{0 \leq j \leq r}^{n \geq 0}$. To compute its eigenvalue, we evaluate at $\bar{\lambda} = \bar{\emptyset}$. This sum equals

$$\frac{(-1)^{n(n+1)/2+rn}}{s_1^n} \sum_{l_1+\dots+l_r=n} \left\{ \prod_{a,b=1}^r \prod_{k=1}^{l_b} (x_a - x_b - (l_a - k + 1)s_2)^{-1} \prod_{b=1}^r \prod_{k=1}^{l_b} ((k-1)s_2 - x_b)^j \right\}.$$

It is straightforward to check that this sum is a rational function in x_a with no poles. Together with the degree estimate for $j \leq r-1$, we see that it is independent of x_a . To compute this constant we let $x_1 \rightarrow \infty$. If $j \leq r-2$, then all summands tend to 0. For $j = r-1$ the only nonzero contribution (equal to $D_{r-1,-n}$) comes from the collection $(l_1, l_2, \dots, l_r) = (n, 0, \dots, 0)$.

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